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A Note on the Assessment of Local Influence in Statistical Models with Equality Constraints

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Relationship has been discussed by Tanaka and Zhang(1999) between the sensitivity analyses based on influence functions and on Cook's local influence, and it has been shown that equivalent results are obtained under general conditions by these analyses in statistical modeling without/with equality constraints. However, a condition implicitly assumed in the proof in Tanaka and Zhang(1999) in the case with equality constraints does not necessarily hold. The present paper gives a complete proof without assuming the condition. Also a formula for the normal curvature is derived for the convenience of practical computation.

keywords: Local influence, influence function, equality constraints

1. INTRODUCTION

There are two major approaches for sensitivity analysis in statistical modeling. One is the influence function approach and the other is the local influence approach. At first the former approach including case deletion is exclusively employed for sensitivity analysis in regression analysis and related methods, and later it spreads in sensitivity analysis in multivariate and other statistical methods (see, e.g., Belsley, Kuh and Welsch, 1980; Cook and Weisberg, 1982). However, after Cook(1986) proposed the latter approach as an alternative methodology to the former approach, it has been applied to various statistical models(Lawrance, 1988; Thomas and Cook, 1989, 1990; Laurent and Cook, 1933; Wang and Lee, 1986; Lesaffre and Verbeke, 1998; Pan, Fang and Rosen, 1997; Kwan and Fung, 1988, Tanaka and Zhang, 1999). Cook(1986) derived the fundamental formulation of local influence in general statistical modeling in the cases where our interest is in all parameters and where it is in a subset of parameters, and discussed in detail the local influence in regression analysis. He assumed full rank models and did not consider models with equality constraints. In multivariate analysis we often meet statistical models where there exist some equality constraints among parameters.

Concerning the local influence in statistical models with equality constraints Wang and Lee(1996) studied the case where we are interested in all parameters, and Kwan and Fung(1998) studied the case where we are interested in a subset of parameters. Tanaka and Zhang(1999) discussed the relationship between the local influence and influence function approaches in general statistical modeling by considering the four cases defined by whether we are interested in all parameters or a subset of parameters and whether there exist equality constraints or not. In the case where we are interested in a subset of parameters and there exist equality constraints, however, a conditions implicitly assumed by Kwan and Fung(1998) is not necessarily satisfied, and Tanaka and Zhang(1999) gave an alternative formulation.

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But, there still exists a possibility, though it becomes smaller than before, that another condition assumed by Tanaka and Zhang(1999) is not satisfied.

In the present paper we give a new formulation in which we do not assume the above mentioned conditions, and show that the relationship of the two approaches discussed by Tanaka and Zhang(1999) holds in general. Also we derive a formula for the normal curvature convenient for practical computation.

2. COOK'S LOCAL INFLUENCE IN STATISTICAL MODELS WITH EQUALITY CONSTRAINTS

Suppose we have a set of n observations and consider a statistical model with parameter vector $\underline{\theta} \in R^m$, where there are r constraints $h_j(\underline{\theta}) = 0$ $(j = 1, \ldots, r)$ among the parameters. Let $L(\underline{\theta})$ be the log likelihood function and $\tilde{\underline{\theta}}$ be the restricted maximum likelihood estimator(RMLE), which is obtained by maximizing $L(\underline{\theta})$ under the r constraints. Define the Lagrangian function by

$$G(\underline{\theta}, \underline{\nu}) = -L(\underline{\theta}) + \underline{h}^T \underline{\nu},$$

where $\underline{\nu}$ is a vector of Lagrange multipliers and $\underline{h}^T = (h_1, \ldots, h_r)$. Then the RMLE $\tilde{\underline{\theta}}$ is obtained by solving the system of likelihood equations

$$\frac{\partial G(\underline{\theta}, \underline{\nu})}{\partial(\underline{\theta}^T, \underline{\nu}^T)} = \underline{0}. \tag{1}$$

Denote the unperturbed weights for n observations by $\underline{\omega}_0 = (1, \ldots, 1)^T$, and consider a perturbation from $\underline{\omega}_0$ to $\underline{\omega} = (\omega_1, \ldots, \omega_n)^T$. Also denote the perturbed log likelihood function by $L(\underline{\theta}|\underline{\omega})$. For simplicity $L(\underline{\theta}) = L(\underline{\theta}|\underline{\omega}_0)$. The RMLE $\tilde{\underline{\theta}}_w$ for the perturbed case is obtained by minimizing

$$G(\underline{\theta}, \underline{\nu}|\underline{\omega}) = -L(\underline{\theta}|\underline{\omega}) + \underline{h}^T \underline{\nu}.$$

In Cook's local influence the amount of change from $\tilde{\underline{\theta}}$ to $\tilde{\underline{\theta}}_w$ is measured with the likelihood displacement

$$D(\underline{\omega}) = 2[L(\tilde{\underline{\theta}}|\underline{\omega}_0) - L(\tilde{\underline{\theta}}_w|\underline{\omega}_0)]$$

and the effect of perturbation is represented by the so-called influence graph

$$\underline{\alpha}(\underline{\omega}) = \begin{pmatrix} \underline{\omega} \\ D(\underline{\omega}) \end{pmatrix}.$$

Cook(1986) focused on the perturbation along a straight line, i.e., $\underline{\omega} = \underline{\omega}_0 + a\underline{d}$, where $||d|| = 1$, and searches for the direction $\underline{d}_{max}$ which has the largest normal curvature at $\underline{\omega}_0$ as the most influential direction. Then he regards the individuals with large absolute elements in $\underline{d}_{max}$ as the influential subset of observations.

Suppose that the parameters are partitioned into $\underline{\theta}^T = (\underline{\theta}_1^T, \underline{\theta}_2^T)$ and that we are interested only in $\underline{\theta}_1$. Then the likelihood displacement is defined by

$$D_s(\underline{\omega}) = 2\left[L(\tilde{\underline{\theta}}_1, \underline{\theta}_2(\tilde{\underline{\theta}}_1)|\underline{\omega}_0) - L(\tilde{\underline{\theta}}_{1w}, \underline{\theta}_2(\tilde{\underline{\theta}}_{1w})|\omega_0)\right],$$

where $\underline{\theta}_2(\underline{\theta}_1)$ is defined as the value of $\underline{\theta}_2$ which minimizes the Lagrangian function for fixed $\underline{\theta}_1$, and $D_s(\underline{\omega})$ plays the similar role as $D(\underline{\omega})$ in evaluating the influence of perturbation.

2.1 Formulation in previous studies

Let $\ddot{G}$ be the second derivative of G with respect to $(\underline{\theta}^T, \underline{\nu}^T)$. Kwan and Fung(1998) partition the extented parameters, i.e., parameters and Lagrange multipliers, into $(\underline{\theta}_1^T, (\underline{\theta}_2^T, \underline{\nu}^T))$ and assume implicitly that the matrix

$$B = \begin{pmatrix} G_{\underline{\theta}_2\underline{\theta}_2} & G_{\underline{\theta}_2\underline{\nu}} \\ G_{\underline{\nu}\underline{\theta}_2} & 0 \end{pmatrix}$$

corresponding to $(\underline{\theta}_2^T, \underline{\nu})$ is not degenerated. In other words, they assume that $(\underline{\theta}_2^T, \underline{\nu})$ can be expressed as a function of $\underline{\theta}_1$, since B is nonsingular in eq.(1). But, as shown in Zhang et al.(1999) there is a possibility that B is degenerated and that the formulation by Kwan and Fung(1998) cannot be used. On the other hand Tanaka and Zhang(1999) partition the constraints into $\underline{h}^T = (\underline{h}_2^T, \underline{h}_1^T)$, where $\underline{h}_2$ is the constraints related to $\underline{\theta}_2$ and $\underline{h}_1$ is the remaining constraints. They partition the Lagrange multipliers as $\underline{\nu}^T = (\underline{\nu}_2^T, \underline{\nu}_1^T)$, and also $\ddot{G}$ into the parts corresponding to $(\underline{\theta}_1^T, (\underline{\theta}_2^T, \underline{\nu}_2^T), \underline{\nu}_1^T)$. In this formulation it is implicitly assumed that

$$Q_{22} = \begin{pmatrix} G_{\underline{\theta}_2\underline{\theta}_2} & G_{\underline{\theta}_2\underline{\nu}_2} \\ G_{\underline{\nu}_2\underline{\theta}_2} & 0 \end{pmatrix},$$

which corresponds to $(\underline{\theta}_2^T, \underline{\nu}_2^T)$, is not degenerated. However, there is a possibility that it does not hold.

Now consider a simple example. Let Σ be a 2×2 covariance matrix and P be the corresponding correlation matrix. Define $\underline{\theta}^T = (\lambda_1, \lambda_2, \underline{\gamma}_1^T, \underline{\gamma}_2^T, \sigma_{11}, \sigma_{22})$, where $P = \Gamma\Lambda\Gamma^T$, $\Gamma = (\underline{\gamma}_1, \underline{\gamma}_2) = (\gamma_{ij})$, σ_{11} and σ_{22} being the diagonal elements of Σ. Constraints are expressed as $\Gamma^T\Gamma = I$, $diag(\Gamma\Lambda\Gamma^T) = I$. Suppose that we are interested in $(\underline{\gamma}_1^T, \underline{\gamma}_2^T)$. Partition the parameters as $(\underline{\theta}_1^T, \underline{\theta}_2^T) = (\underline{\gamma}_1^T, \underline{\gamma}_2^T; \lambda_1, \lambda_2, \sigma_{11}, \sigma_{22})$ and the constraints as $\underline{h}^T = (\underline{h}_2^T, \underline{h}_1^T)$, where $\underline{h}_2^T = (\lambda_1\gamma_{11}^2 + \lambda_2\gamma_{12}^2 - 1, \lambda_1\gamma_{21}^2 + \lambda_2\gamma_{22}^2 - 1)$, $\underline{h}_1^T = (\underline{\gamma}_1^T\underline{\gamma}_1 - 1, 2\underline{\gamma}_1^T\underline{\gamma}_2, \underline{\gamma}_2^T\underline{\gamma}_2 - 1)$.

$$G_{\underline{\nu}_2\underline{\theta}_2} = \frac{\partial \underline{h}_2}{\partial \underline{\theta}_2^T} = \begin{pmatrix} \gamma_{11}^2 & \gamma_{12}^2 & 0 & 0 \\ \gamma_{21}^2 & \gamma_{22}^2 & 0 & 0 \end{pmatrix}$$

Obviously the eigenvectors of a 2×2 correlation matrix is given by

$$\Gamma = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore

$$G_{\underline{\nu}_2\underline{\theta}_2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Hence, obviously Q_{22} is degenerated.

2.2 New formulation

In Tanaka and Zhang(1999) $\underline{h}_2$ is selected as the constraints containing $\underline{\theta}_2$. But, it does not guarantee that Q_{22} is not degenerated. That is, the corresponding $(\underline{\theta}_2^T, \underline{\nu}_2^T)$ cannot necessarily be expressed by using $(\underline{\theta}_1^T, \underline{\nu}_1^T)$. Therefore, we select $\underline{h}_2$ as the maximum constraints where $(\underline{\theta}_2^T, \underline{\nu}_2^T)$ can be expressed by using $(\underline{\theta}_1^T, \underline{\nu}_1^T)$ and partition $\underline{h}$ into $\underline{h}^T = (\underline{h}_2^T, \underline{h}_1^T)$. In other words, we select $\underline{h}_2$ in such a way that Q_{22} is a nondegenerated principal submatrix with the maximum size of

$$\begin{pmatrix} G_{\underline{\theta}_2\underline{\theta}_2} & G_{\underline{\theta}_2\underline{\nu}_2} & G_{\underline{\theta}_2\underline{\nu}_1} \\ G_{\underline{\nu}_2\underline{\theta}_2} & 0 & 0 \\ G_{\underline{\nu}_1\underline{\theta}_2} & 0 & 0 \end{pmatrix}$$

Then we can proceed the discussion as in Tanaka and Zhang(1999). Let us partition $\ddot{G}$ into the part of $(\underline{\theta}_1^T, (\underline{\theta}_2^T, \underline{\nu}_2^T), \underline{\nu}_1^T)$ as

$$\ddot{G} = \begin{pmatrix} Q_{11} & Q_{12} & H_{1\underline{\theta}_1} \\ Q_{21} & Q_{22} & H_{12} \\ H_{1\underline{\theta}_1}^T & H_{12}^T & 0 \end{pmatrix}$$

where

$$Q_{11} = G_{\underline{\theta}_1\underline{\theta}_1}, \quad Q_{12} = Q_{21}^T = (G_{\underline{\theta}_1\underline{\theta}_2}, G_{\underline{\theta}_1\underline{\nu}_2}),$$

$$Q_{22} = \begin{pmatrix} G_{\underline{\theta}_2\underline{\theta}_2} & G_{\underline{\theta}_2\underline{\nu}_2} \\ G_{\underline{\nu}_2\underline{\theta}_2} & 0 \end{pmatrix},$$

$$H_{1\underline{\theta}_1}^T = G_{\underline{\nu}_1\underline{\theta}_1} = \frac{\partial \underline{h}_1}{\partial \underline{\theta}_1^T},$$

$$H_{12}^T = \begin{pmatrix} \frac{\partial \underline{h}_1}{\partial \underline{\theta}_2^T} & \frac{\partial \underline{h}_1}{\partial \underline{\nu}_2^T} \end{pmatrix} = (H_{1\underline{\theta}_2}^T \quad 0).$$

Then the normal curvature in the case we are interested in $\underline{\theta}_1$ is given by

$$C_{\underline{d}}(\underline{\theta}_1) = 2\left| \underline{d}^T \frac{\partial \tilde{\underline{\theta}}_{1w}^T}{\partial \underline{w}} Q_{11\cdot 2} \frac{\partial \tilde{\underline{\theta}}_{1w}}{\partial \underline{w}^T} \underline{d} \right|, \tag{2}$$

where $Q_{11\cdot 2} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$.

Next, let us try to derive a consistent estimate V_{11} for $\mathrm{acov}(\tilde{\underline{\theta}}_1)$ to discuss the relationship between the Cook's local influence approach and the influence function approach. As discussed by Tanaka and Zhang(1999) V_{11} is obtained as the upper left part of the inverse of matrix $\ddot{G}$ or equivalently as the same part of

$$\begin{pmatrix} Q_{11} & H_{1\underline{\theta}_1} & Q_{12} \\ H_{1\underline{\theta}_1}^T & 0 & H_{12}^T \\ Q_{21} & H_{12} & Q_{22} \end{pmatrix}.$$

Hence, it is obtained as the upper left part of

$$\left[\begin{pmatrix} Q_{11} & H_{1\underline{\theta}_1} \\ H_{1\underline{\theta}_1}^T & 0 \end{pmatrix} - \begin{pmatrix} Q_{12} \\ H_{12}^T \end{pmatrix} Q_{22}^{-1} \begin{pmatrix} Q_{21} & H_{12} \end{pmatrix} \right]^{-1}$$

$$= \begin{pmatrix} Q_{11\cdot 2} & H_{1\underline{\theta}_1} - Q_{12}Q_{22}^{-1}H_{12} \\ H_{1\underline{\theta}_1}^T - H_{12}^TQ_{22}^{-1}Q_{21} & -H_{12}^TQ_{22}^{-1}H_{12} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} Q_{11\cdot 2} & H_{1\underline{\theta}_1} - Q_{12}Q_{22}^{-1}H_{12} \\ H_{1\underline{\theta}_1}^T - H_{12}^TQ_{22}^{-1}Q_{21} & 0 \end{pmatrix}^{-1}. \tag{3}$$

It can be proved that $Q_{11\cdot 2}$ is a g-inverse of V_{11}, and therefore, following the discussion of Tanaka and Zhang(1999) the two approaches provide the equivalent result. The last equality of eq.(3) can be verified as follows.

Let $\underline{h}_1 = (h_{11}, \cdots, h_{1t})$. Since Q_{22} is the maximum nondegenerated submatrix, it follows for any $i, j = 1, \cdots, t$,

$$\begin{vmatrix} G_{\underline{\theta}_2\underline{\theta}_2} & G_{\underline{\theta}_2\underline{\nu}_2} & \frac{\partial h_{1j}}{\partial \underline{\theta}_2} \\ G_{\underline{\nu}_2\underline{\theta}_2} & 0 & 0 \\ \frac{\partial h_{1i}}{\partial \underline{\theta}_2^T} & 0 & 0 \end{vmatrix} = 0.$$

$$|Q_{22}| \left| 0 - \begin{pmatrix} \frac{\partial h_{1i}}{\partial \underline{\theta}_2^T} & 0 \end{pmatrix} Q_{22}^{-1} \begin{pmatrix} \frac{\partial h_{1j}}{\partial \underline{\theta}_2^T} & 0 \end{pmatrix}^T \right| = 0$$

$$\therefore \begin{pmatrix} \frac{\partial h_{1i}}{\partial \underline{\theta}_2^T} & 0 \end{pmatrix} Q_{22}^{-1} \begin{pmatrix} \frac{\partial h_{1j}}{\partial \underline{\theta}_2^T} & 0 \end{pmatrix}^T = 0.$$

Therefore, since i and j are arbitrary, we finally obtain

$$H_{12}^TQ_{22}^{-1}H_{12} = 0.$$

3. COMPUTATION OF NORMAL CURVATURE

Eq.(2) in the previous section is a formula derived for studying the relationship with the influence function approach. It is not convenient for actual computation. In this section we derive a more convenient formula.

Rearrange the order of the elements of $\ddot{G}$ so that they follows the natural ordering of $(\underline{\theta}_1^T, \underline{\theta}_2^T, \underline{\nu}_1^T, \underline{\nu}_2^T)$, and denote the obtained matrix by $\ddot{G}_0$. Partition the parameters into $(\underline{\theta}_1^T, (\underline{\theta}_2^T, \underline{\nu}^T))$ and express the corresponding partitioned matrix by

$$\ddot{G}_0 = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.$$

Then the following theorem holds.

Theorem. Suppose we are interested in $\underline{\theta}_1$ of the partitioned parameter vector $\underline{\theta}^T = (\underline{\theta}_1^T, \underline{\theta}_2^T)$. Then the normal curvature is expressed as

$$C_{\underline{d}}(\underline{\theta}_1) = 2\left|\underline{d}^T \Delta_{\underline{\theta}}^T \left(G_0^{\underline{\theta\theta}} - \begin{bmatrix} 0 & 0 \\ 0 & G_{22}^{\underline{\theta}_2\underline{\theta}_2} \end{bmatrix}\right) \Delta_{\underline{\theta}} \underline{d}\right|,$$

where $G_0^{\underline{\theta\theta}}$ is a submatrix of

$$\ddot{G}_0^{-1} = \begin{pmatrix} G_0^{\underline{\theta\theta}} & G_0^{\underline{\theta\nu}} \\ G_0^{\underline{\nu\theta}} & G_0^{\underline{\nu\nu}} \end{pmatrix},$$

and $G_{22}^{\underline{\theta}_2\underline{\theta}_2}$ is a submatrix of

$$G_{22}^{+} = \begin{pmatrix} G_{22}^{\underline{\theta}_2\underline{\theta}_2} & G_{22}^{\underline{\theta}_2\underline{\nu}_1} & G_{22}^{\underline{\theta}_2\underline{\nu}_2} \\ G_{22}^{\underline{\nu}_1\underline{\theta}_2} & G_{22}^{\underline{\nu}_1\underline{\nu}_1} & G_{22}^{\underline{\nu}_1\underline{\nu}_2} \\ G_{22}^{\underline{\nu}_2\underline{\theta}_2} & G_{22}^{\underline{\nu}_2\underline{\nu}_1} & G_{22}^{\underline{\nu}_2\underline{\nu}_2} \end{pmatrix},$$

where G_{22}^{+} indicates the Moore-Penrose inverse of G_{22}.

Proof. First we show

$$C_{\underline{d}}(\underline{\theta}_1) = 2\left|\underline{d}^T \Delta_{\underline{\theta}}^T \left(G^{\underline{\theta\theta}} - \begin{bmatrix} 0 & 0 \\ 0 & Q_{22}^{\underline{\theta}_2\underline{\theta}_2} \end{bmatrix}\right) \Delta_{\underline{\theta}} \underline{d}\right|, \qquad (*)$$

then show

$$G_0^{\underline{\theta\theta}} = G^{\underline{\theta\theta}} \quad , \quad G_{22}^{\underline{\theta}_2\underline{\theta}_2} = Q_{22}^{\underline{\theta}_2\underline{\theta}_2}, \qquad (**)$$

where $G^{\underline{\theta\theta}}$ and $Q_{22}^{\underline{\theta}_2\underline{\theta}_2}$ is the upper left parts of $\ddot{G}^{-1}$ and Q_{22}^{-1}, respectively. Consider eq.(2) in section 2. Differentiate both sides of the system of likelihood equations

$$\frac{\partial G(\underline{\theta}, \underline{\nu}|\underline{\omega})}{\partial(\underline{\theta}^T, \underline{\nu}^T)^T} = \underline{0},$$

with respect to $\underline{\omega}^T$ and solve the resulting equation for $\partial(\underline{\theta}^T, \underline{\nu}^T)^T / \partial\underline{\omega}^T$, then

$$\frac{\partial(\underline{\theta}^T, \underline{\nu}^T)^T}{\partial\underline{\omega}^T} = -\ddot{G}^{-1}\Delta, \qquad (4)$$

where

$$\Delta = \frac{\partial^2 G(\underline{\theta}, \underline{\nu}|\underline{\omega})}{\partial(\underline{\theta}^T, \underline{\nu}^T)^T \partial\underline{\omega}^T} = \begin{pmatrix} \frac{\partial^2 G(\underline{\theta}, \underline{\nu}|\underline{\omega})}{\partial\underline{\theta}\partial\underline{\omega}^T} \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta_{\underline{\theta}} \\ 0 \end{pmatrix}. \qquad (5)$$

Substituting eq.(4) into eq.(2) and taking into account the relation

$$\frac{\partial\hat{\underline{\theta}}_{1\underline{w}}}{\partial\underline{w}^T}=(I\ \ 0\ \ 0)\frac{\partial(\underline{\theta}_1^T,(\underline{\theta}_2^T,\underline{\nu}_2^T),\underline{\nu}_1^T)^T}{\partial\underline{w}^T},$$

we obtain

$$C_{\underline{d}}(\underline{\theta}_1)=2\left|\underline{d}^T\Delta^T\ddot{G}^{-1}\begin{pmatrix}Q_{11\cdot2}&0&0\\0&0&0\\0&0&0\end{pmatrix}\ddot{G}^{-1}\Delta\underline{d}\right|, \tag{6}$$

where $\ddot{G}$ is partitioned as

$$\ddot{G}=\begin{pmatrix}Q_{11}&Q_{12}&H_{1\underline{\theta}_1}\\Q_{21}&Q_{22}&H_{12}\\H_{1\underline{\theta}_1}^T&H_{12}^T&0\end{pmatrix}$$

corresponding to partition $(\underline{\theta}_1^T,(\underline{\theta}_2^T,\underline{\nu}_2^T),\underline{\nu}_1^T)$.

Then, to derive $\ddot{G}^{-1}$ we consider

$$\ddot{G}_1^{-1}=\begin{pmatrix}Q_{11}&H_{1\underline{\theta}_1}&Q_{12}\\H_{1\underline{\theta}}^T&0&H_{12}^T\\Q_{21}&H_{12}&Q_{22}\end{pmatrix}^{-1}.$$

Since the $(\underline{\theta}_1^T,\underline{\nu}_1^T)$ part of $\ddot{G}_1^{-1}$ can be obtained as

$$\begin{aligned}&\left[\begin{pmatrix}Q_{11}&H_{1\underline{\theta}_1}\\H_{1\underline{\theta}_1}^T&0\end{pmatrix}-\begin{pmatrix}Q_{12}\\H_{12}^T\end{pmatrix}Q_{22}^{-1}(Q_{21}\ \ H_{12})\right]^{-1}\\=&\begin{pmatrix}Q_{11\cdot2}&H_{1\underline{\theta}_1}-Q_{12}Q_{22}^{-1}H_{12}\\H_{1\underline{\theta}_1}^T-H_{12}^TQ_{22}^{-1}Q_{21}&0\end{pmatrix}^{-1}\\=&\begin{pmatrix}V_{11}&V_{12}\\V_{21}&V_{22}\end{pmatrix},\end{aligned} \tag{7}$$

then, using the formula for the inverse of partitioned matrices (see, e.g., Schott, 1997, Theorem 7.1), $\ddot{G}_1^{-1}$ can be expressed as

$$\begin{pmatrix}\begin{pmatrix}V_{11}&V_{12}\\V_{21}&V_{22}\end{pmatrix}&-\begin{pmatrix}V_{11}&V_{12}\\V_{21}&V_{22}\end{pmatrix}\begin{pmatrix}Q_{12}\\H_{12}^T\end{pmatrix}Q_{22}^{-1}\\-Q_{22}^{-1}(Q_{21}\ \ H_{12})\begin{pmatrix}V_{11}&V_{12}\\V_{21}&V_{22}\end{pmatrix}&Q_{22}^{-1}+Q_{22}^{-1}(Q_{21}\ \ H_{12})\begin{pmatrix}V_{11}&V_{12}\\V_{21}&V_{22}\end{pmatrix}\begin{pmatrix}Q_{12}\\H_{12}^T\end{pmatrix}Q_{22}^{-1}\end{pmatrix}.$$

Therefore $\ddot{G}^{-1}$ becomes a symmetric matrix whose lower half part is given by

$$\begin{pmatrix}V_{11}&&\\-Q_{22}^{-1}(Q_{21}V_{11}+H_{12}V_{21})&Q_{22}^{-1}+Q_{22}^{-1}(Q_{21}\ \ H_{12})\begin{pmatrix}V_{11}&V_{12}\\V_{21}&V_{22}\end{pmatrix}\begin{pmatrix}Q_{12}\\H_{12}^T\end{pmatrix}Q_{22}^{-1}&\\V_{21}&-(V_{21}Q_{12}+V_{22}H_{12}^T)Q_{22}^{-1}&V_{22}\end{pmatrix}, \tag{8}$$

and the upper left part corresponding to $(\underline{\theta}_1^T,(\underline{\theta}_2^T,\underline{\nu}_2^T))$ of

$$\ddot{G}^{-1}\begin{pmatrix}Q_{11\cdot2}&0&0\\0&0&0\\0&0&0\end{pmatrix}\ddot{G}^{-1}$$

can be expressed as

$$\begin{pmatrix} V_{11}Q_{11\cdot 2}V_{11} & -V_{11}Q_{11\cdot 2}(V_{11}Q_{12}+V_{12}H_{12}^T)Q_{22}^{-1} \\ -Q_{22}^{-1}(Q_{21}V_{11}+H_{12}V_{21})Q_{11\cdot 2}V_{11} & Q_{22}^{-1}(Q_{21}V_{11}+H_{12}V_{21})Q_{11\cdot 2}(V_{11}Q_{12}+V_{12}H_{12}^T)Q_{22}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} V_{11} & -V_{11}Q_{12}Q_{22}^{-1} \\ -Q_{22}^{-1}Q_{21}V_{11} & Q_{22}^{-1}Q_{21}V_{11}Q_{12}Q_{22}^{-1}+Q_{22}^{-1}H_{12}V_{21}Q_{11\cdot 2}V_{12}H_{12}^TQ_{22}^{-1} \end{pmatrix} \tag{9}$$

To derive eq.(9) we use the relations $V_{11}Q_{11\cdot 2}V_{11} = V_{11}$ and $V_{21}Q_{11\cdot 2}V_{11} = 0$. These relations can be easily derived from eq.(7).

From section 2.2, $H_{12}^TQ_{22}^{-1}H_{12} = 0$. Using the relations

$$Q_{22}^{-1} = \begin{pmatrix} Q_{22}^{\underline{\theta}_2\underline{\theta}_2} & Q_{22}^{\underline{\theta}_2\underline{\nu}_2} \\ Q_{22}^{\underline{\nu}_2\underline{\theta}_2} & Q_{22}^{\underline{\nu}_2\underline{\nu}_2} \end{pmatrix}, \quad H_{12}^T = (H_{1\underline{\theta}_2}^T \quad 0),$$

it follows

$$H_{1\underline{\theta}_2}^T Q_{22}^{\underline{\theta}_2\underline{\theta}_2} H_{1\underline{\theta}_2} = 0.$$

As $Q_{22}^{\underline{\theta}_2\underline{\theta}_2}$ is nonnegative definite from the theory of RMLE, $Q_{22}^{\underline{\theta}_2\underline{\theta}_2}H_{1\underline{\theta}_2} = 0$. In other words, the part of $Q_{22}^{-1}H_{12}$ which corresponds to $\underline{\theta}_2$ is zero, i.e.,

$$Q_{22}^{-1}H_{12} = (0 \quad *)^T. \tag{10}$$

Comparing the upper left part of $\ddot{G}^{-1}$ in eq.(8) and eq.(9), we can find that the upper left part corresponding to $\underline{\theta}$ of

$$\ddot{G}^{-1}\begin{pmatrix} Q_{11\cdot 2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ddot{G}^{-1}$$

is expressed as

$$G^{\underline{\theta}\underline{\theta}} - \begin{pmatrix} 0 & 0 \\ 0 & Q_{22}^{\underline{\theta}_2\underline{\theta}_2} \end{pmatrix}.$$

Therefore, from eq.(5) and eq.(6), we obtain eq.(*).

We shall proceed to the proof of eq.(**).

It is obvious that the first equation of eq.(**) holds.

In the second equation $G_{22}^{\underline{\theta}_2\underline{\theta}_2}$ is the upper left part of either side matrix of

$$\begin{pmatrix} G_{\underline{\theta}_2\underline{\theta}_2} & G_{\underline{\theta}_2\underline{\nu}_1} & G_{\underline{\theta}_2\underline{\nu}_2} \\ G_{\underline{\nu}_1\underline{\theta}_2} & 0 & 0 \\ G_{\underline{\nu}_2\underline{\theta}_2} & 0 & 0 \end{pmatrix}^- = \begin{pmatrix} Q_{22} & H_{12} \\ H_{12}^T & 0 \end{pmatrix}^-.$$

From the definition of Q_{22} in section 2.2 it is obvious that the column space of H_{12} is contained in the column space of Q_{22}. Thus, from Harville(1997, Theorem 9.6.1) the above matrix can be rewritten by

$$\begin{pmatrix} Q_{22} & H_{12} \\ H_{12}^T & 0 \end{pmatrix}^-$$

$$= \begin{pmatrix} Q_{22}^{-1} - Q_{22}^{-1}H_{12}(H_{12}^TQ_{22}^{-1}H_{12})^-H_{12}^TQ_{22}^{-1} & Q_{22}^{-1}H_{12}(H_{12}^TQ_{22}^{-1}H_{12})^{-1} \\ (H_{12}^TQ_{22}^{-1}H_{12})^-H_{12}^TQ_{22}^{-1} & -(H_{12}^TQ_{22}^{-1}H_{12})^- \end{pmatrix}$$

Taking into account the relation $Q_{22}^{-1}H_{12} = (0 \quad *)^T$ we can verify that the upper left corner of the above matrix is equal to the upper left corner of Q_{22}^{-1}. Therefore,

$$G_{22}^{\underline{\theta}_2\underline{\theta}_2} = Q_{22}^{\underline{\theta}_2\underline{\theta}_2},$$

where $G_{22}^{\theta_2\theta_2}$ is a submatrix of G_{22}^{-}. As it is obvious that this submatrix does not depend on the choice of g-inverse, we can use the Moore-Penrose inverse in place of a general g-inverse. Q.E.D.

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