

Superconductivity Proximity Effect in Inhomogeneous Media

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SYNOPSIS

The superconductivity proximity effect in the dirty limit is revisited and equations and boundary conditions are obtained for the case of inhomogeneous media with continuously varying characteristic parameters.

I. INTRODUCTION

In microstructures including superconducting materials, the proximity effect^{1,2,3} is expected to play a significant role to determine various properties of these structures which will be used as a part of electronic devices.⁴ The simplest case of the proximity effect may be the one in the dirty limit where the mean free path of electrons is much smaller than other characteristic lengths of the system. The system is then characterized by the parameters such as the effective interaction between electrons, the local density of states, and the diffusion coefficient.

The proximity effect in the dirty limit has been first formulated by de Gennes^{1,2} on the basis of the one-particle correlation function of electrons in the normal state. He has given the equations and boundary conditions governing the integral kernel of the Gor'kov equation^{5,3} when the system is composed of several homogeneous domains with different parameters.¹ The proximity effect in this case has been analyzed by several authors.⁶

In structures composed of different materials, there may exist the cases where the junction is not abrupt and values of parameters change continuously from one side to another with the scale length comparable with the characteristic length of the kernel. It may also be possible to realize such junctions intentionally. In order to analyze these cases, we have to properly take into account the inhomogeneity of media. The purpose of this paper is to revisit the superconductivity proximity effect in the dirty limit and obtain the formulation applicable to inhomogeneous as well as homogeneous media.

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II. FORMULATION

Near the critical point of second order transition between the superconducting and the normal phases, the pair potential $\Delta(\mathbf{r})$ satisfies the Gor'kov equation^{5,3}

$$\Delta(\mathbf{r}) = V(\mathbf{r})k_B T \sum_{\omega} \int Q(\mathbf{r}, \mathbf{r}'; \omega) \Delta(\mathbf{r}') d\mathbf{r}'. \quad (2.1)$$

Here $V(\mathbf{r})$ is the effective interaction between electrons, $\omega = \pi(2n+1)k_B T/\hbar$ is the Matsubara frequency, and the kernel $Q(\mathbf{r}, \mathbf{r}'; \omega)$ is given by the Green's function for normal electrons $G^0(\mathbf{r}, \mathbf{r}'; \omega)$ on the Fermi surface as

$$Q(\mathbf{r}, \mathbf{r}'; \omega) = G^0(\mathbf{r}, \mathbf{r}'; -\omega) G^0(\mathbf{r}, \mathbf{r}'; \omega). \quad (2.2)$$

Following de Gennes,^{1,2} we introduce the one-electron correlation function $g(\mathbf{r}, \mathbf{r}'; t)$ defined by

$$g(\mathbf{r}, \mathbf{r}'; t) = \overline{\langle m | K(t)^\dagger \delta(\mathbf{r} - \mathbf{r}(t)) \delta(\mathbf{r}' - \mathbf{r}(0)) K(0) | m \rangle}, \quad (2.3)$$

where $K(t)$ is the operator of complex-conjugation (time reversal) and the average is taken over the one-electron states $|m\rangle$ on the Fermi surface. The kernel $Q(\mathbf{r}, \mathbf{r}'; \omega)$ is expressed by $g(\mathbf{r}, \mathbf{r}'; t)$ as

$$Q(\mathbf{r}, \mathbf{r}'; \omega) = \frac{2\pi}{\hbar} \int_0^\infty dt g(\mathbf{r}, \mathbf{r}'; t) \exp(-2|\omega|t). \quad (2.4)$$

The correlation function $g(\mathbf{r}, \mathbf{r}'; t)$ satisfies the initial condition

$$g(\mathbf{r}, \mathbf{r}'; t=0) = N(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}'), \quad (2.5)$$

where $N(\mathbf{r})$ is the local density of states at the Fermi level.¹

In the dirty limit, electrons follow the diffusion process⁷ and the time scale of the change in one-electron correlation function is much longer than that of collisions with impurities. We take the time increment δt which is much smaller than the former and much larger than the latter, and denote the probability to make a transition of $\delta \mathbf{r}$ at \mathbf{r} in the time interval δt by $w(\mathbf{r}, \delta \mathbf{r})$. The time-evolution of the operator of $g(\mathbf{r}, \mathbf{r}'; t)$ may then be expressed as

$$g(\mathbf{r}, \mathbf{r}'; t + \delta t) = \int g(\mathbf{r} - \delta \mathbf{r}, \mathbf{r}'; t) w(\mathbf{r} - \delta \mathbf{r}, \delta \mathbf{r}) \exp(a\mathbf{A}(\mathbf{r}) \cdot \delta \mathbf{r}) d(\delta \mathbf{r}). \quad (2.6)$$

Here $\mathbf{A}(\mathbf{r})$ is the vector potential and the factor $\exp(a\mathbf{A}(\mathbf{r}) \cdot \delta \mathbf{r})$ with

$$a = 2ie/\hbar c \quad (2.7)$$

comes from the time evolution of the operator $K(t)$: We assume that the spatial variation of the vector potential $\mathbf{A}(\mathbf{r})$ is small and semiclassical treatment of the operator $K(t)$ is possible.^{5,2}

We expand $g(\mathbf{r}-\delta\mathbf{r}, \mathbf{r}'; t)$ and $w(\mathbf{r}-\delta\mathbf{r}, \delta\mathbf{r})$ around \mathbf{r} and $\exp(a\mathbf{A}(\mathbf{r})\cdot\delta\mathbf{r})$ with respect to $\delta\mathbf{r}$, retain terms up to the second order, and obtain the Fokker-Planck equation⁷ for our system as

$$\frac{\partial}{\partial t}g(\mathbf{r}, \mathbf{r}'; t) = (\nabla - a\mathbf{A}(\mathbf{r})) \cdot [-\mathbf{F}(\mathbf{r}) + (\nabla - a\mathbf{A}(\mathbf{r})) \cdot \mathbf{D}(\mathbf{r})]g(\mathbf{r}, \mathbf{r}'; t). \quad (2.8)$$

Here the 'friction' coefficient $\mathbf{F}(\mathbf{r})$ and the diffusion tensor $\mathbf{D}(\mathbf{r})$ are defined by

$$\mathbf{F}(\mathbf{r}) = \frac{\langle \delta\mathbf{r} \rangle}{\delta t}, \quad (2.9)$$

$$\mathbf{D}(\mathbf{r}) = \frac{1}{2} \frac{\langle \delta\mathbf{r}\delta\mathbf{r} \rangle}{\delta t}, \quad (2.10)$$

with

$$\langle \delta\mathbf{r} \rangle = \int \delta\mathbf{r}w(\mathbf{r}, \Delta)d(\delta\mathbf{r}), \quad (2.11)$$

$$\langle \delta\mathbf{r}\delta\mathbf{r} \rangle = \int \delta\mathbf{r}\delta\mathbf{r}w(\mathbf{r}, \Delta)d(\delta\mathbf{r}). \quad (2.12)$$

With the increase of the time t , the distribution function $g(\mathbf{r}, \mathbf{r}'; t)$ (as a function of \mathbf{r}) will relax to the one-particle distribution at thermal equilibrium and become stationary. Thus the right-hand side of our Fokker-Planck equation vanishes when

$$g(\mathbf{r}, \mathbf{r}'; t) \propto N(\mathbf{r}) \exp(a\mathbf{A}(\mathbf{r}) \cdot \mathbf{r}). \quad (2.13)$$

This condition gives a relation between the friction coefficient and the diffusion tensor

$$\mathbf{F}(\mathbf{r}) = \frac{1}{N(\mathbf{r})} \nabla \cdot \mathbf{D}(\mathbf{r})N(\mathbf{r}). \quad (2.14)$$

The above relation between $\mathbf{F}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ allows us to rewrite eq.(2.8) into the form

$$\frac{\partial}{\partial t}g(\mathbf{r}, \mathbf{r}'; t) = (\nabla - a\mathbf{A}(\mathbf{r})) \cdot N(\mathbf{r})\mathbf{D}(\mathbf{r}) \cdot (\nabla - a\mathbf{A}(\mathbf{r})) \frac{g(\mathbf{r}, \mathbf{r}'; t)}{N(\mathbf{r})} \quad (2.15)$$

or

$$\frac{\partial}{\partial t} \frac{g(\mathbf{r}, \mathbf{r}'; t)}{N(\mathbf{r})N(\mathbf{r}')} = \frac{1}{N(\mathbf{r})} (\nabla - a\mathbf{A}(\mathbf{r})) \cdot N(\mathbf{r})\mathbf{D}(\mathbf{r}) \cdot (\nabla - a\mathbf{A}(\mathbf{r})) \frac{g(\mathbf{r}, \mathbf{r}'; t)}{N(\mathbf{r})N(\mathbf{r}')}. \quad (2.16)$$

Eqs.(2.4), (2.5), and (2.16) form the fundamental equations to describe the proximity effect in the dirty limit.

From eq.(2.16), we see that the correlation function satisfies the conditions at interfaces

$$\frac{g(\mathbf{r}, \mathbf{r}'; t)}{N(\mathbf{r})N(\mathbf{r}')} = \text{continuous} \quad (2.17)$$

and

$$\mathbf{n} \cdot N(\mathbf{r})D(\mathbf{r}) \cdot (\nabla - a\mathbf{A}(\mathbf{r})) \frac{g(\mathbf{r}, \mathbf{r}'; t)}{N(\mathbf{r})N(\mathbf{r}')} = \text{continuous}, \quad (2.18)$$

where \mathbf{n} is the unit vector normal to the interface. At the surface of the system, we have

$$\mathbf{n} \cdot N(\mathbf{r})D(\mathbf{r}) \cdot (\nabla - a\mathbf{A}(\mathbf{r})) \frac{g(\mathbf{r}, \mathbf{r}'; t)}{N(\mathbf{r})N(\mathbf{r}')} = 0. \quad (2.19)$$

We note that, due to eq.(2.1), $\Delta(\mathbf{r})/N(\mathbf{r})V(\mathbf{r})$ satisfies the same boundary conditions as the kernel or the one-particle correlation function (as a function of \mathbf{r}). Thus we have the boundary conditions for the pair potential

$$\frac{\Delta}{N(\mathbf{r})V(\mathbf{r})} = \text{continuous} \quad (2.20)$$

and

$$\mathbf{n} \cdot N(\mathbf{r})D(\mathbf{r}) \cdot (\nabla - a\mathbf{A}(\mathbf{r})) \frac{\Delta}{N(\mathbf{r})V(\mathbf{r})} = \begin{cases} \text{continuous (at interfaces),} \\ 0 \text{ (at surfaces).} \end{cases} \quad (2.21)$$

When the system is homogeneous and isotropic, the friction coefficient vanishes and the diffusion tensor reduces to the unit tensor multiplied by a scalar constant D . In this case, our equation and boundary conditions reproduce the original ones given by de Gennes,^{1,2}

$$\frac{\partial}{\partial t} g(\mathbf{r}, \mathbf{r}'; t) = D(\nabla - a\mathbf{A}(\mathbf{r}))^2 g(\mathbf{r}, \mathbf{r}'; t) \quad (2.22)$$

and

$$D\mathbf{n} \cdot (\nabla - a\mathbf{A}(\mathbf{r})) \frac{\Delta}{V} = 0 \text{ at surfaces.} \quad (2.23)$$

When the system is composed of several homogeneous isotropic domains, eq.(2.16) in each domain can be rewritten as

$$\frac{\partial}{\partial t} g(\mathbf{r}, \mathbf{r}'; t) = D(\mathbf{r})(\nabla - a\mathbf{A}(\mathbf{r}))^2 g(\mathbf{r}, \mathbf{r}'; t), \quad (2.24)$$

where $D(\mathbf{r})$ is constant in each domain, and we can solve the problem applying the boundary conditions

$$\frac{\Delta}{NV} = \text{continuous} \quad (2.25)$$

$$D(\mathbf{r})\mathbf{n} \cdot (\nabla - a\mathbf{A}(\mathbf{r}))\frac{\Delta}{V} = \text{continuous (at interfaces) or 0 (at surfaces)} \quad (2.26)$$

for $\Delta(\mathbf{r})$ at all the interfaces and surfaces. These results are also due to de Gennes.¹

When the properties of the system changes continuously, however, we have to resort to eq.(2.16) and the boundary conditions (2.20) and (2.21), instead of above equations and boundary conditions for special cases. We also note that our results, though general, are derived in a rather simple way by somewhat different considerations on diffusion processes from those due to de Gennes.¹

III. METHOD OF SOLUTION

A useful method to solve eq.(2.1) is to use the eigenfunctions $\phi_n(\mathbf{r})$ of the operator \mathcal{L} defined by

$$\mathcal{L} = -\frac{\hbar}{N(\mathbf{r})}(\nabla - a\mathbf{A}(\mathbf{r})) \cdot D(\mathbf{r})N(\mathbf{r}) \cdot (\nabla - a\mathbf{A}(\mathbf{r})). \quad (3.1)$$

We denote the eigenvalues by ε_n ;

$$[\mathcal{L} - \varepsilon_n]\phi_n(\mathbf{r}) = 0. \quad (3.2)$$

The boundary conditions for eigenfunctions are that $\phi_n(\mathbf{r})$ and $\mathbf{n} \cdot N(\mathbf{r})D(\mathbf{r}) \cdot \nabla\phi_n(\mathbf{r})$ are continuous at interfaces and the latter vanishes at surfaces.

So as to make the operator \mathcal{L} hermitian, we here define the inner product of functions Ψ and Φ as

$$(\Psi, \Phi) = \int d\mathbf{r}N(\mathbf{r})\Psi(\mathbf{r})\Phi(\mathbf{r}). \quad (3.3)$$

We may then assume that the eigenfunctions of the operator \mathcal{L} form a complete orthogonal set with respect to this inner product.

We expand the pair potential into these eigenfunctions as

$$\frac{\Delta(\mathbf{r})}{N(\mathbf{r})V(\mathbf{r})} = \sum_n a_n \phi_n(\mathbf{r}). \quad (3.4)$$

Eq.(2.1) thus reduces to

$$\sum_{n'} C_{nn'} a_{n'} = 0, \quad (3.5)$$

where

$$C_{nn'} = \delta_{nn'} - A_n \frac{\int d\mathbf{r} N(\mathbf{r}) \phi_n(\mathbf{r}) V(\mathbf{r}) N(\mathbf{r}) \phi_{n'}(\mathbf{r})}{[\int d\mathbf{r} N(\mathbf{r}) \phi_n(\mathbf{r})^2 \int d\mathbf{r}' N(\mathbf{r}') \phi_{n'}(\mathbf{r}')^2]^{\frac{1}{2}}}, \quad (3.6)$$

$$A_n = \ln(1.13\Theta_D/T) + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{\epsilon_n}{4\pi k_B T}\right), \quad (3.7)$$

and $\psi(x)$ is the digamma function. We have applied the standard cutoff procedure³ to the summation over ω denoting the Debye temperature by Θ_D . The critical parameters are determined by the condition that eq.(3.5) has a nontrivial solution for the pair potential or

$$\det | C_{nn'} | = 0. \quad (3.8)$$

This method has been first used by de Gennes for the homogeneous case² and has also been applied to piecewise homogeneous cases. The above formulation is the generalization to inhomogeneous cases based on the results of preceding section.

IV. CONCLUSION

We have obtained the equations and boundary conditions for the integral kernel of the Gor'kov equation which are applicable in inhomogeneous media. The derivation is based on a general consideration of the diffusion process of electrons in the dirty limit. The method to determine the critical parameters is also reformulated accordingly.

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