MEMOIRS OF THE SCHOOL OF ENGINEERING, OKAYAMA UNIVERSITY, Vol. 1, No. 1, MARCH, 1966

Approximate Solution of Nonlinear Oscillatory Circuits (I)

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(Received November 20, 1965)

The well-known steady state solution of nonlinear oscillatory circuits based on the use of Fourier series, the so-called harmonic balance, is a powerful method because of its wide applicability. Yet, this method has not only difficulties to solve transcendental equations or algebraic equations of higher orders, but gives no transient state solution. The harmonic approximation and the extended harmonic approximation defined in this report are linearization methods which give approximations of steady state and transient state simultaneously.

Furthermore the method enables an unification and extension of miscellaneous linearization methods.

§ I. Introduction

There are many analytic methods of solution for nonlinear oscillatory circuits. The so-called harmonic balance[1], based on the Fourier expansion, is a powerful method of wide applicability. This method requires a process to solve transcendental equations or algebraic equations of higher orders and yet gives no transient state solution.

The present report propose a new conception of harmonic approximation (or harmonically approximated system) of a nonlinear oscillatory circuits. The harmonic approximation is based on the harmonic balance and defined as a system which is linear conditionally, and the solution has considerable accuracy of approximation to the original system. The extended harmonic approximation is also a conditionally linear system which has much remarkable accuracy of approximation to the transient state of the original system.

The equation of harmonic approximation can be solved simply by the use of an analogue computer giving simultaneously steady state and transient state solutions.

The report (I) deals mainly with steady state.

§ II. Definition of harmonic approximation

In this report, oscillatory circuits which have periodic steady state solution are treated. In general, equation of nonlinear circuits contains a nonlinear term $f(x, \dot{x}, \ddot{x}, \cdots)$. Now we deal with nonlinear term of the form $f(x, \dot{x})$, because the calculations of the general form $f(x, \dot{x}, \ddot{x}, \cdots)$ can be made in almost the similar ways. This nonlinear term f is approximated by a linear function $f_h(x, \dot{x})$ according to the following process. Since it is assumed that x is periodic for steady state,

$$x=x_0+\sum_{n=1}^{n}x_n,\quad x_n=x_{nm}\sin(n\omega t-\theta_n).$$

$f(x, \dot{x})$ is also a periodic function and expressed by the following Fourier expansion:

$$f(x, \dot{x})=A_0+\sum_{n=1}^{\infty}A_n\sin(n\omega t-\theta_n)+\sum_{n=1}^{\infty}B_n\cos(n\omega t-\theta_n),$$

where

$$A_0=\frac{1}{2\pi}\int_0^{2\pi}fd(\omega t),$$

$$A_n=\frac{1}{\pi}\int_0^{2\pi}f\sin(n\omega t-\theta_n)d(\omega t),$$

$$B_n=\frac{1}{\pi}\int_0^{2\pi}f\cos(n\omega t-\theta_n)d(\omega t),$$

$$f=f\left\{x_0+\sum_{n=1}^{n}x_{nm}\sin(n\omega t-\theta_n),\ \sum_{n=1}^{n}n\omega x_{nm}\cos(n\omega t-\theta_n)\right\}.$$

Corresponding to the expansion of x, we take the first n terms of f, and define it as the (0, 1, ... n) harmonic approximation f_h of f. f_h can be expressed in terms of x_n as follows:

$$f_h=\frac{A_0}{x_0}x_0+\sum_{n=1}^{n}\frac{A_n}{x_{nm}}x_n+\sum_{n=1}^{n}\frac{B_n}{n\omega x_{nm}}\dot{x}_n$$

$$=\alpha_0x_0+\sum_{n=1}^{n}(\alpha_nx_n+\frac{\beta_n}{n\omega}\dot{x}_n),$$

where

$$\alpha_0\equiv A_0/x_0,\quad \alpha_n\equiv A_n/x_{nm},\quad \beta_n\equiv B_n/x_{nm}.$$

The system having f_h instead of f in the original system is linear conditionally and has a steady state solution which is identical up to the n-th harmonics with the original nonlinear system. This system is defined as the "(0, 1,$\cdots n$) harmonic approximation" (or harmonically approximated system) of the original system. The case where $n=1$, we name it simply the "harmonic approximation" of the original system. As to the transient solution, it will be shown later that when the nonlinearity is not very great, the harmonic approximation also gives a good approximation for transient state.

Example 1

We take a ferro-resonant circuit represented by the following equations:

$$\begin{cases} N\dot{\phi}+Ri+\int idt/C=\sqrt{2}E\sin\omega t \\ i=f(\phi)=a\phi+b\phi^3. \end{cases}$$

It is known that the magnetic flux ϕ has a wave form very nearly sinusoidal. Accordingly we assume $\phi=\phi_m\sin(\omega t-\theta)$. Substituting ϕ into $i=f(\phi)$, we expand i into Fourier series and from the series we take up the same fundamental frequency term as the flux. Then, we obtain the harmonic approximation i_h for nonlinear element i:

$$\begin{aligned} i&=f(\phi)=f(\phi_m\sin(\omega t-\theta)) \\ &=A_1\sin(\omega t-\theta)+A_3\sin(3\omega t-\theta_3)+\cdots \\ i_h&=A_1\sin(\omega t-\theta)=A_1\phi/\phi_m=\alpha\phi, \end{aligned}$$

where

$$\begin{aligned} \alpha&=\frac{1}{\phi_m\pi}\int_0^{2\pi}f(\phi_m\sin(\omega t-\theta))\sin(\omega t-\theta)d(\omega t) \\ &=a+\frac{3}{4}b\phi_m^2. \end{aligned}$$

Therefore, the linear integro-differential equation for the harmonically approximated system becomes as

$$N\dot{\phi}+R\alpha\phi+\frac{1}{C}\int\alpha\phi dt=\sqrt{2}E\sin\omega t.$$

Example 2

We take a unsymmetrical nonlinear system repesented by the following equations:

$$\begin{cases} m\ddot{x}+c\dot{x}+f(x)=p\sin\omega t \\ f(x)=ax+bx^2. \end{cases}$$

It is assumed that the solution consists of two terms, namely oscillatory and non-oscillatory term:

$x=x_0+x_1$, $x_1=x_m\sin(\omega t-\theta)$, x_0: a term independent of time. Upon substitution of x into $f(x)$, f is expanded into Fourier series, and from the series we take up the same fundamental frequency and non-oscillatory term as x. Thus, f_h becomes

$$\begin{aligned} f_h&=A_0+A_1\sin(\omega t-\theta)=\frac{A_0}{x_0}x_0+\frac{A_1}{x_m}x_1 \\ &=\alpha_0x_0+\alpha_1x_1, \end{aligned}$$

where

$$\begin{cases} \alpha_0=\dfrac{1}{x_0 2\pi}\displaystyle\int_0^{2\pi}fd(\omega t)=a+bx_0+\frac{b}{2x_0}x_m^2 \\ \alpha_1=\dfrac{1}{x_m\pi}\displaystyle\int_0^{2\pi}f\sin(\omega t-\theta)d(\omega t)=a+2bx_0 \end{cases}$$

$$f=f(x_0+x_m\sin(\omega t-\theta)).$$

f_h is the (0, 1) harmonic approximation. Therefore the (0, 1) harmonic approximation is expressed by a linear equation as

$$m\ddot{x}+c\dot{x}+\alpha_0x_0+\alpha_1x_1=p\sin\omega t,$$

and according to the principle of harmonic balance we obtain

$$m\ddot{x}_1+c\dot{x}_1+\alpha_1x_1=p\sin wt,\quad \alpha_0x_0=0. \quad (1)$$

Example 3

We take a subharmonic oscillation represented by the following equations:

$$\begin{cases} \ddot{x}+c\dot{x}+f(x)=p\cos 3t \\ f(x)=x^3. \end{cases}$$

It is assumed that a subharmonics ocour. Hence, we assume the solution as

$$\begin{aligned} &x=x_1+x_3,\quad x_1=x_{1m}\cos(t-\theta_1), \\ &x_3=x_{3m}\cos(3t-\theta_3). \end{aligned}$$

Substituting x into $f(x)$, we expand f into Fourier series. We take up only the terms of angular frequency 1, 3, thus, the (1, 3) harmonic approximation f_h becomes

$$\begin{aligned} f_h&=A_1\cos(t-\theta_1)+B_1\sin(t-\theta_1) \\ &\quad+A_3\cos(3t-\theta_3)+B_3\sin(3t-\theta_3) \\ &=\frac{A_1}{x_{1m}}x_1-\frac{B_1}{x_{1m}}\dot{x}_1+\frac{A_3}{x_{3m}}x_3-\frac{B_3}{3x_{3m}}\dot{x}_3 \\ &=\alpha_1x_1+\beta_1\dot{x}_1+\alpha_3x_3+\frac{\beta_3}{3}\dot{x}_3, \end{aligned}$$

where

$$\begin{cases} \alpha_1 = \dfrac{1}{x_{1m}\pi}\displaystyle\int_0^{2\pi} f\cos(t-\theta_1)dt \\ \quad = \dfrac{3}{4}x_{1m}^2 + \dfrac{3}{2}x_{3m}^2 + \dfrac{3}{4}x_{1m}x_{3m}\cos(3\theta_1-\theta_3) \\ \beta_1 = \dfrac{-1}{x_{3m}\pi}\displaystyle\int_0^{2\pi} f\sin(t-\theta_1)dt \\ \quad = \dfrac{3}{4}x_{1m}x_{3m}\sin(3\theta_1-\theta_3) \\ \alpha_3 = \dfrac{1}{x_{3m}\pi}\displaystyle\int_0^{2\pi} f\cos(3t-\theta_3)dt \\ \quad = \dfrac{3}{4}x_{3m}^2 + \dfrac{3}{2}x_{1m}^2 + \dfrac{x_{1m}^2}{4x_{3m}}\cos(3\theta_1-\theta_3) \\ \beta_3 = \dfrac{-1}{x_{3m}\pi}\displaystyle\int_0^{2\pi} f\sin(3t-\theta_3)dt \\ \quad = \dfrac{-x_{1m}^3}{4x_{3m}}\sin(3\theta_1-\theta_3) \end{cases}$$

The (1, 3) harmonically approximated system is given by following linear equations:

$$\begin{cases} \ddot{x}_1 + (c+\beta_1)\dot{x}_1 + \alpha_1 x_1 = 0 \\ \ddot{x}_3 + (c+\dfrac{\beta_3}{3})\dot{x} + \alpha_3 x_3 = p\cos 3t. \end{cases}$$

Example 4

Oscillatory circuit containing a hysteresis element is treated:

$$\begin{cases} N\dot{\phi} + Ri + \displaystyle\int idt/C = \sqrt{2}E\sin\omega t \\ i = f(\phi). \end{cases}$$

As in example 1, we assume a solution in the form $\phi = \phi_m \sin(\omega t-\theta)$. The harmonic approximation of nonlinear element $f(\phi)$ becomes

$$i_h = A_1\sin(\omega t-\theta) + B_1\mathrm{con}(wt-\theta) = \alpha\phi + \frac{\beta}{\omega}\dot{\phi},$$

where

$$\alpha = \frac{1}{\phi_m\pi}\int_0^{2\pi} f\sin(\omega t-\theta)d(\omega t),$$

$$\beta = \frac{1}{\phi_m\pi}\int_0^{2\pi} f\cos(\omega t-\theta)d(\omega t)$$

$$f = f(\ _m\sin(\omega t-\theta)).$$

Therefore, the harmonically approximated system is given by the following linear equation:

$$N\dot{\phi} + R(\alpha\phi + \frac{\beta}{\omega}\dot{\phi}) + \frac{1}{C}\int(\alpha\phi + \frac{\beta}{\omega}\dot{\phi})dt$$
$$= \sqrt{2}E\sin\omega t. \qquad (2)$$

§ III. Method of solution of the harmonically approximated system

While the harmonic approximation is expressed formally by a linear equation, its coefficients are not constants. They must satisfy certain relations with the amplitudes x_{nm} of x_n. The solution requires therefore cumbersome calculations and sometimes insolvable by ordinary analytic method. But, there is a great merit in harmonic approximation that it permits the use of an analogue computer in simple form. Owing to its linearity, an analogue computer with only linear elements is sufficient to solve the equation of harmonic approximation. The troubles associated with nonlinear elements (multipliers, function generators etc.) are avoided. The solution by an analogue computer includes both steady state and transient state. The details of operation are explained in the following examples, and the problem of the accuracy of the transient solution will be treated in part (II) of the report.

Example 1

Symmetrical nonlinear element

$$\begin{cases} m\ddot{x} + c\dot{x} + f(x) = p\sin\omega t \\ f(x) = kx + bx^3 \end{cases}$$

Assuming $x = x_m\sin(\omega t-\theta)$, we obtain the harmonically approximated system as

$$m\ddot{x} + c\dot{x} + \alpha x = p\sin\omega t, \qquad (3)$$

where

$$\alpha = \frac{1}{x_m\pi}\int_0^{2\pi} f(x_m\sin(\omega t-\theta))\sin(\omega t-\theta)d(\omega t)$$
$$= k + \frac{3}{4}bx_m^2.$$

The solution of the equation (3) by analogue computer is as follows. First we draw a graph, Fig. 1 of α for x_m from the above relation. The computer diagram of the equation (3) is Fig. 2. We assume certain value for x_m, then, we calculate α for this x_m from the graph Fig. 1, and we put the value into the potentiometer α of Fig. 2. The operation is begun, and we observe whether the amplitude of steady solution agrees with that of the first assumption x_m. If it does not agree, we take another value for x_m, and repeat such operations, untill we obtain the agreement of the amplitude of the solution with that of the first assumption.

This cut and try process requires only to change some (in this case one) potentiometer. Therefore, after a few trials, we can obtain a sufficiently precise solution. Now, to show an numerical example, we assume as follows;

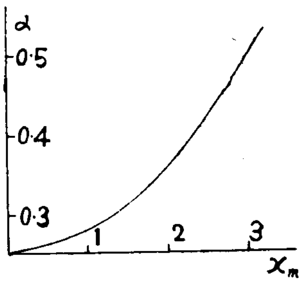


Fig. 1. Relation between x_m and linearization parameter α.

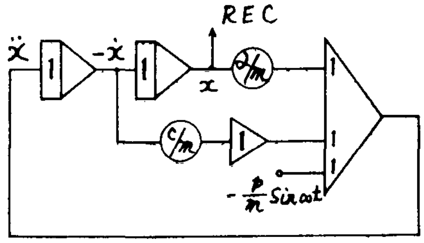


Fig. 2. Computer diagram of the equation (3).

$$\frac{c}{\sqrt{km}}=0.17,\ b/k=0.133,\ p/k=0.8,$$

$$u=\omega/\sqrt{km},$$

$$\alpha_t=\sqrt{k/m}\ \text{(time scale factor)}$$

Figs. 3~4 show some of the results. Fig. 3 shows amplitudes of steady state. In Fig. 3, the dotted line gives the exact solution by a computer with multipliers and the solid line gives the harmonic approximation. In Fig. 4, the actual results by a computer are given, that is, *a* is the exact solution with multipliers and *b* is the harmonic approximation.

From Figs. 3~4, we know that the harmonic approximation becomes the approximation of the exact solution even at transient state.

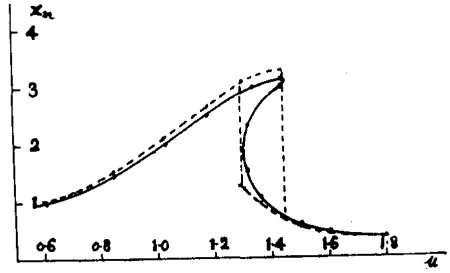


Fig. 3. Conparison between the exact solution and the harmonic approximation.

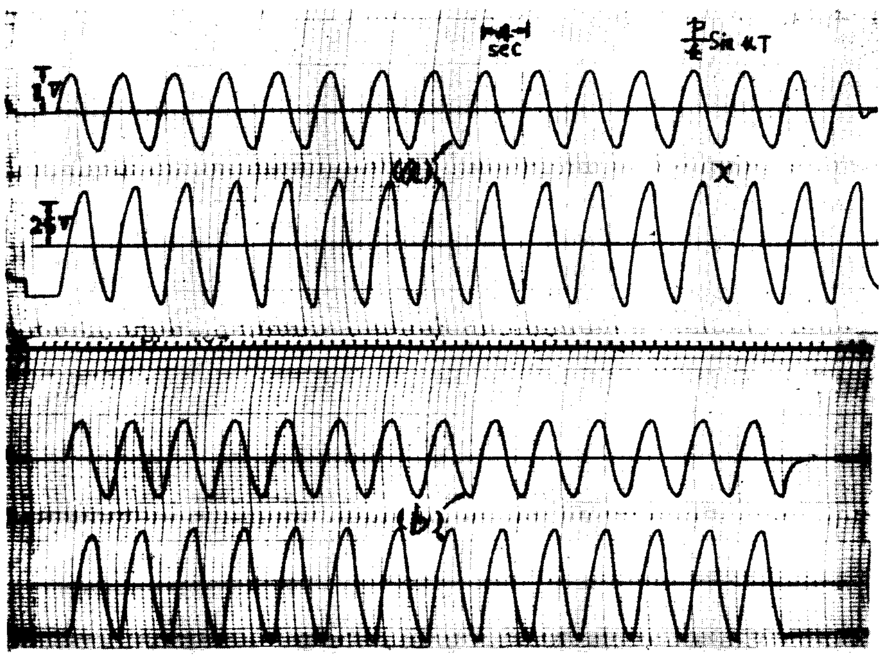


Fig. 4. Comparison between the exact solution and the harmonic approximation.

Example 2

Unsymmetrical nonlinear element

$$\begin{cases} m\ddot{x}+c\dot{x}+f(x)=p\sin\omega t \\ f(x)=kx+bx^2 \end{cases}$$

It is assumed that the solution is

$$x=x_0+x_1=x_0+x_m\sin(\omega t-\theta).$$

Then, according to the equation (1) and $\ddot{x}=\dot{x}=0$, the (0, 1) harmonically approximated system is expressed as follows:

$$m\ddot{x}+c\dot{x}+\alpha_1 x=p\sin\omega t+\alpha_1 x_0. \quad (4)$$

Refering $\alpha_0 x_0 = 0$ and the above example 2, we obtain

$$(k + bx_0 + bx_m^2/2x_0)x_0 = 0.$$

Therefore, we get x_0 for x_m and α_1 from $\alpha_1 = k + 2bx_0$ and Fig. 5 successively. The computer diagram of the equaticn (4) is Fig. 6.

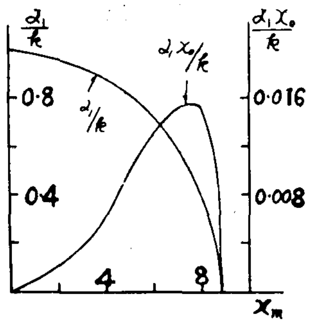


Fig. 5. Relations between x_m and parameters α_1/k, $\alpha_1 x_0/k$.

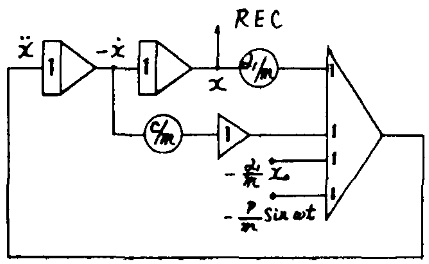


Fig. 6. Coemputer diagram of the equation (4).

First we assume certain value for x_m, then, we calculate α_1 and $\alpha_1 x_0$ for this x_m from the graph Fig. 5 and we put those values into the potentiometer α_1, $\alpha_1 x_0$ of Fig. 6. By cut and try process as before, we repeat the operations, untill we obtain the agreement of the amplitude of the solution with that of the first assumption. Now to show an numerical example, we assume as follows:

$$\frac{c}{\sqrt{km}} = 0.17, \quad b/k = 0.08, \quad p/k = 1.2,$$

$$u = \omega/\sqrt{k/m}, \quad \alpha_t = \sqrt{k/m}.$$

Figs. 7~8 show some of the results. Fig. 7 shows a steady solution. In Fig. 7, the sign × shows the exact solution by a computer by multipliers and the sign ○ shows the (0, 1) harmonic approximation. In Fig. 8, the actual results with a computer are given, that is, a is the exact solution with multipliers and b is the (0, 1) harmonic approximation. We know that the (0, 1) harmonic approximations in Figs. 7~8 give good approximation of the exact solution.

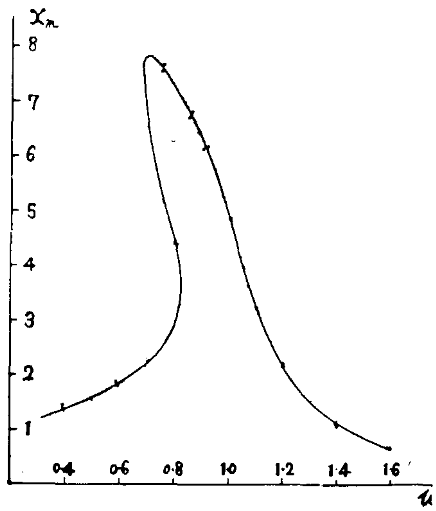


Fig. 7. Comparison between the exact solution and the harmonic approximation.

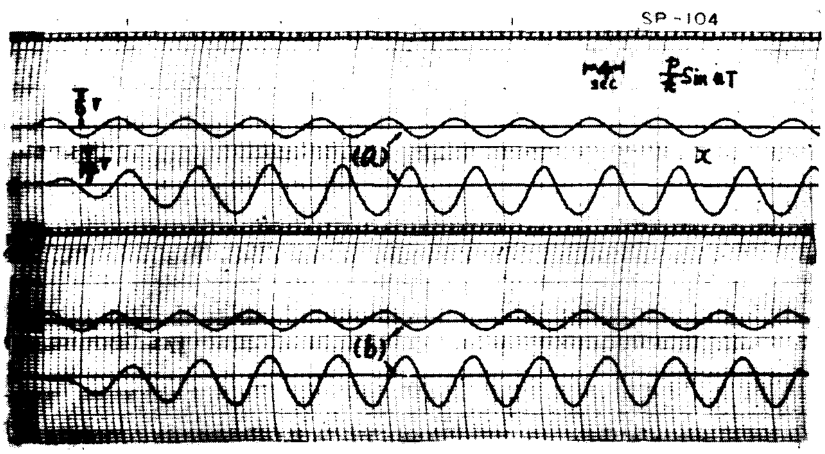


Fig. 8. Comparison between the exact solution and the harmonic approximation.

Example 3

Hysteresis element

We again deal with the above example 4. Thus, the equation (2) gives the harmonically approximated system. First we draw a graph, Fig. 9 of α, β for ϕ_m from hysteresis loops of the iron-core. The computer diagram of the equation (2) is Fig. 10. We assume certain value for ϕ_m, then, we calculate α, β for this ϕ_m from Fig. 9. By cut and try process as before, we repeat the operations, untill we obtain the agreement of the amplitude of the solution with that of the first assumption. To show an numerical example, we assume as follows:

$$R=1.12,\ C=40.8\times10^{-6},\ N=242,$$
$$\alpha_t=200.$$

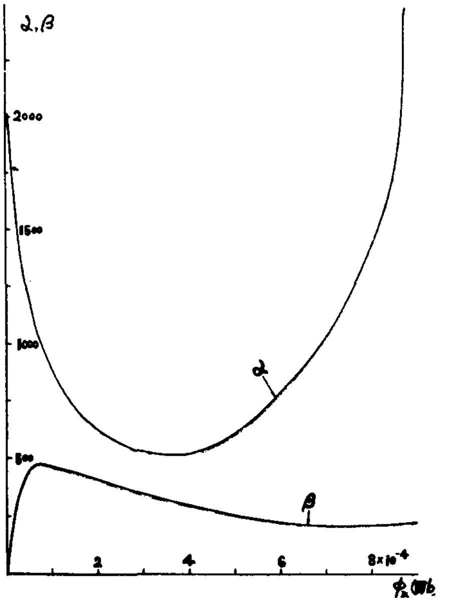


Fig. 9. Relations between ϕ_m and linearization parameters α, β.

Figs. 11~12 show some of the results. Fig. 11 shows a steady solution. In Fig. 11, the dotted line gives the experimental solution with a synchroscope and the solid line gives the harmonic approximation. In Fig. 12, the actual results are given, that is, *a* is the exact solution and *b* is the harmonic approximation. We know that the harmonic approximations in Figs. 11~12 give good approximations of the experimental solution.

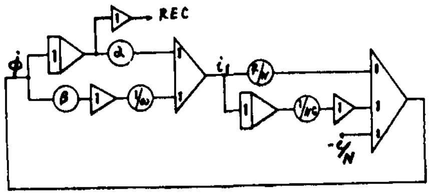


Fig. 10. Computer diagram of the equation (2).

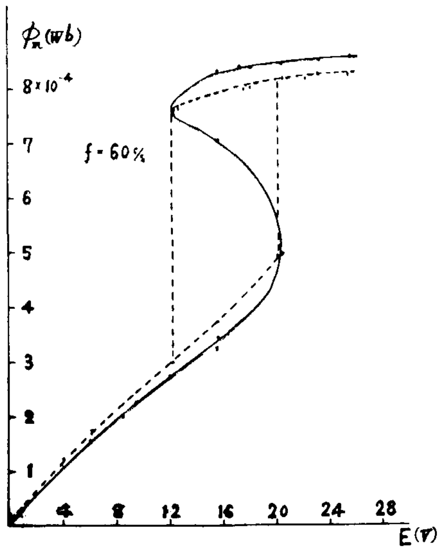
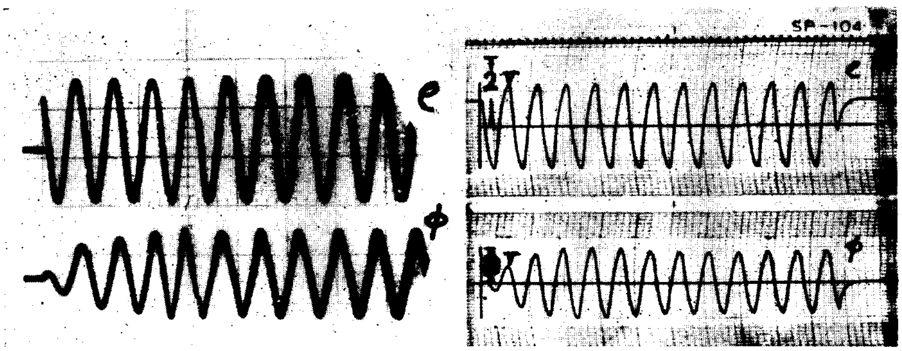


Fig. 11. Comparison between the exact solution and the harmonic approximation.



(a) (b)

Fig. 12. Comparison between the exact solution and the harmonic approximation.

§ IV. Applications to autonomous systems

If there are periodic solutions, the harmonic approximation is also applied to autonomous systems. The periodic solutions of autonomous system appear in the self-excited oscillation and the conservative system. For example, the van der Pol equation is treated:

$$\ddot{x}+\varepsilon(-1+x^2)\dot{x}+x=0.$$

Assuming $x=x_m \sin\omega t$ for the self-excited oscillation, we obtain the harmonic approximation $f_h=-\varepsilon\dot{x}+\varepsilon x_m^2\dot{x}/4$ for the nonlinear element $f(x,\dot{x})=\varepsilon(-1+x^2)\dot{x}$. The harmonically approximated system becomes

$$\ddot{x}+\varepsilon(-1+\frac{1}{4}x_m^2)\dot{x}+x=0.$$

To obtain the periodic solution of this system, we must take as follows:

$$-1+\frac{1}{4}x_m^2=0,\quad \omega=1,\quad \therefore\ x_m=2,\quad \omega=1$$

That is, the harmonic approximation in steady state is $x=2\sin t$. The damped oscillation will be treated in part (II) of the report.

§ V. Conclusions

The essential points of the study are written in the following items:

(1) The harmonic approximation of nonlinear element, and the harmonically approximated system of nonlinear system are defined.

(2) The linearization of nonlinear term is first performed. Then, the nonlinear system is transformed into the harmonically approximated system, which is expressed by a linear integro-differential equation.

(3) The harmonic approximation is similar to the so-called equivalent linearization. However, there is a difference in those linearizations that the harmonic approximation could be approached up to the n-th harmonics of the original system.

(4) The calculation of the harmonic approximation with an analogue computer reduces to the operations of solution of a linear integro-differential equation with constant coefficients. Therefore, both approximate solutions of steady and transient state are simply obtained.

(5) The limit of the problems to be solved by the harmonic approximation and the degree of approximation in steady state theoreticaly agree with those of the so-called harmonic balance method.

(6) The harmonic approximation is also applied for the self-excited oscillation and the conservative system in autonomous system.

(7) The harmonic approximation in steady state is the best approximation of nonlinear element by the method of least squares. That is,

$$\int_0^{2\pi}\left\{f(x,\ \dot{x})-f_h\right\}^2 d(\omega t)=min.$$

This equation is simply obtained by the property of Fourier series.

Acknowledgment

The authors wish to express particular thanks to prolessor Sigenori Hayashi, at the university of Kyoto.

Reference

1) C. Hayashi: *Nonlinear Oscillations in Physical Systems.* Mcgraw-Hill (1964) 28.