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A Note on the Optimal Assignment of Facilities to Locations by Branch and Bound

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The problem of the optimal assignment of facilities to locaticns has becn solved by J. W. Gavett and Normann V. Plyter[1]. In their paper the branch and bound technique[2] is used and for this purpose the distance matrix is transformed into the matrix whose diagonal compoents are zero. The purpose of this paper is to avoid such a transformation and the store of the distance matrix into the computer.

§ 1. The problem is called the 'assigment of facilities to locations' and the general statement of this problem is as follows. There are n fixed locations to which n facilities are to be assigned. Just only one facility may be assigned to each location.

A distance between the locations is associated with each pair of locations and an index of the traffic intensity between the facilities is associated with each pair of facilities. The cost of assigning a pair of facilities to a pair of locations is the product of the location distance by the facility traffic intensity. The cost of the total assignment is the sum of these products for all location-facility pairs in the assignment. If the cost is the minimum then this assignment is called the optimal assignment.

J. W. Gavett and Norman V. Plyter* have solved such a problem with the branch and bound technique. But they transformed the cost matrix into the matrix whose diagonal components are zero and moreover all components of the cost matrix must be stored into the computer. But if n≧10 this method is inappropriate to our computer NEAC-2203 (memory 2K). Thus in this paper we shall show the method by which we can avoid the above difficultis**.

§ 2. First of all we shall show the example which J. W. Gavett and others used in their paper.

Let B (i, j) be the distance between location i and j. Then the distance matrix is given by

i \ j	a_1	a_2	a_3	a_4
a_1	×	6	7	2
a_2	6	×	5	6
a_3	7	5	×	1
a_4	2	6	1	×

Next let A (k, l) be the rate of traffic flow between the facility k and l. Then the traffic intensity matrix is given by

k \ l	1	2	3	4
1	×	10	20	5
2	18	×	9	4
3	5	6	×	8
4	8	0	15	×

Now let the facilities be 1, ⋯, n and the locations be $a_1, \cdots, a_n$. Then the facility-pairs are (1, 2), ⋯, (1, n), (2, 3), ⋯ (n-1, n) and the traffic intensity from the facility i to j is denoted by A(i, j). Thus $A'(i, j)=A(i, j)+A(j, i)$ is the rate of the traffic flow between the facility i and j. Now we rearrange these A′ (i, j) in ascending order (lowest to highest) and we put them $A_1, \cdots, A_N$ $(N=\binom{n}{2})$. Similarly as above the location pairs are $(a_1, a_2), \cdots (a_1, a_n), \cdots, (a_{n-1}, a_n)$ and the distance between the location a_i and a_j is denoted by B (i, j). Then we rearrange these B (i, j) in descending order (highest to lowest) and we put them $B_1, \cdots, B_N$. Moreover we put $B_{ij}=B_i-B_j$. In this way we have $A_1=A'(2, 4)=4$, $A_2=A'(1, 4)=13$, $A_3=A'(2, 3)=15$, $A_4=A'(3, 4)=23$, $A_5=A'(1, 3)=25$, $A_6=A'(1, 2)=28$ and $B_1=B(1, 3)=7$, $B_2=B(1, 2)=6$, $B_3=B(2, 4)=6$, $F_4=B(2, 3)=5$, $B_5=B(1, 4)=2$, $B_6=B(3, 4)=1$.

* See the reference 1).

** Our mcthod is analogous to J. W. Gavett's method except the above mentioned point.

Next we associate the following quantity with the correspondence $B_i \to A_j$;

(1) if $i < j$ then $A_i B_{i+1,i} + A_{i+1} B_{i+2,i+2} + \cdots\cdots + A_j B_{i,j}$,

(2) if $i = j$ or $i = j+1$ then 0,

(3) if $i > j+1$ then $A_i B_{i,i+1} + A_{i-1} B_{i-1,i} + \cdots + A_{j+1} B_{j+1,j+2} + A_j B_{j+1,j+1}$.

These quantities are denoted by $(B_i \to A_j)$ and we put $\theta(i,j) = \min_{k \neq i}(B_i \to A_k) + \min_{l \neq j}(B_l \to A_j)$.

§ **3.** In the first place we choose larger θ (i, j) between θ (1, 1) and θ (N, N). In our example

$$\theta(1,1) = (B_1 \to A_2) + (B_2 \to A_1) = A_2B_{12} + A_1B_{21} + 0 = A_2B_{12} + A_1B_{21} = 13\times(7-6) + 4\times(6-7) = 13-4 = 9,$$

$$\theta(6,6) = (B_6 \to A_5) + (B_5 \to A_6) = A_6B_{56} + A_5B_{65} + 0 = A_6B_{56} + A_5B_{65} = 28\times(2-1) + 25\times(1-2) = 28-25 = 3$$

and $\theta(1,1)$ is larger than $\theta(6,6)$. Hence $B_1 = B(1,3)$ corresponds to $A_1 = A'(2,4)$.

Now let [X] be the set of all assignments. Then the lower bound of [X] is $\sum_{i=1}^{N} A_iB_i = \sum_{i=1}^{6} A_iB_i = 889$. Next we can consider the following two sets. One is the set of all assignments such that $B_1 = B(1,3)$ does not correspond to $A_1 = A'(2,4)$ and the other is the set of all assignments such that $B_1 = B(1,3)$ corresponds to $A_1 = A'(2,4)$. We denote them by $[\overline{B_1 \to A_1}]$ and $[B_1 \to A_1]$ according to J. W. Gavett and others.

The lower bound of $[\overline{B_1 \to A_1}]$ is

$\sum A_iB_i + \max\{\theta(1,1) + \theta(N,N)\} = 389 + 9 = 398$ and the lower bound of $[B_1 \to A_1]$ is computed as follows:

First of all $B_3 = B(2,4)$ must correspond to $A_5 = A'(1,3)$ since $B_1 = B(1,3)$ corresponds to $A_1 = A'(2,4)$. Thus $(B_3 \to A_5) = A_5B_{35} + A_4B_{54} + A_3B_{43} = 25\times(5-2) + 23\times(2-5) + 15\times(5-6) = 16$. Hence the lower bound of $[B_1 \to A_1]$ is $389 + 16 = 405$. But since 405 is larger than 398, the following branching commences at node $[\overline{B_1 \to A_1}]$ and we replace $(B_1 \to A_i)$ $(i = 2, \cdots, N) = (A_iB_{1i} + A_{i-1}B_{i,i-1} + \cdots + A_2B_{32} + A_1B_{21}$ with $(B_1 \to A_i) = A_iB_{1i} + \cdots + A_1B_{21} - (A_2B_{12} + A_1B_{21})$.

§ **4.** Next we select the maximum θ(i, j) among $\theta(6,6)$, $\theta(2,1)$ and $\theta(1,2)$. Then

$$\theta(6,6) = A_6B_{56} + A_5B_{65} = 3,$$

$$\theta(2,1) = (B_2 \to A_2) + (B_3 \to A_1) = 0 + A_2B_{23} + A_1B_{32} = 13\times0 + 4\times0 = 0,$$

$$\theta(1,2) = (B_1 \to A_3) + (B_2 \to A_2) = (A_3B_{13} + A_2B_{32} + A_1B_{21}) - (A_2B_{12} + A_1B_{21}) + 0 = A_3B_{13} + A_2B_{31} = 15\times(7-6) + 13\times(6-7) = 15-13 = 2$$

and $\theta(6,6) = 3$ is the largest of all. Thus $B_6 = B(3,4)$ corresponds to $A_6 = A'(1,2)$ and the branches of $[\overline{B_1 \to A_1}]$ are $[\overline{B_6 \to A_6}]$ and $[B_6 \to A_6]$.

Now the lower bound of $[\overline{B_6 \to A_6}]$ is $398 + 3 = 401$ and the lower bound of $[B_6 \to A_6]$ is computed as follows;

$B_2 = B(1,2)$ must correspond to $A_4 = A'(3,4)$ since $B_6 = B(3,4)$ corresponds to $A_6 = A'(1,2)$. Hence $(B_2 \to A_4) = A_4B_{24} + A_3B_{43} + A_2B_{32} = 23\times(6-5) + 15\times(5-6) + 13\times(6-6) = 8$ and the lower bound of $[B_6 \to A_6]$ is $398 + 8 = 406$ but 406 is larger than 401. Thus the following branching commences at node $[\overline{B_6 \to A_6}]$ and in order to continue this process we replace $(B_j \to A_6)$ $(j = 1, \cdots, 5) = (A_6B_{j6} + A_5B_{65} + \cdots + A_jB_{j+1,j})$ with $(B_j \to A_6) = A_6B_{j6} + \cdots + A_jB_{j+1,j} - (A_6B_{56} + A_5B_{65})$.

§ **5.** Similarly as above we select the maximum θ(i, j) among $\theta(1,2)$, $\theta(2,1)$, $\theta(5,6)$ and $\theta(6,5)$. Then

$$\theta(1,2) = (B_1 \to A_3) + (B_2 \to A_2) = A_3B_{13} + A_2B_{31} = 15\times(7-6) + 13\times(6-7) = 2,$$

$$\theta(2,1) = (B_3 \to A_1) + (B_2 \to A_2) = A_2B_{23} + A_1B_{32} = 13\times0 + 4\times0 = 0,$$

$$\theta(5,6) = (B_5 \to A_5) + (B_4 \to A_6) = (A_6B_{46} + A_5B_{65} + A_4B_{54}) - (A_6B_{56} + A_5B_{65}) = A_6B_{45} + A_4B_{54} = 28\times(5-2) + 23\times(2-5) = 15,$$

$$\theta(6,5) = (B_6 \to A_4) + (B_5 \to A_5) = A_5B_{56} + A_4B_{65} = 25\times(2-1) + 23\times(1-2) = 25-23 = 2$$

and $\theta(5,6) = 15$ is maximum. Thus $B_5 = B(1,4)$ corresponds to $A_6 = A'(1,2)$ and $[\overline{B_5 \to A_6}]$ and $[B_5 \to A_6]$ are the branches of $[\overline{B_6 \to A_6}]$.

Now the lower bound of $[\overline{B_5 \to A_6}]$ is $401 + 15 = 416$ and the lower bound of $[B_5 \to A_6]$ is $401 + 0 = 401$ since $B_4 = B(2,3)$ corresponds to $A_4 = A'(3,4)$ and $(B_4 \to A_4) = 0$. Thus the following branching commences at node $[B_5 \to A_6]$ since 401 is smaller than 416.

§ **6.** From now on we assume that $(B_4 \to A_j)(j \neq 4)$, $(B_5 \to A_j)(j \neq 6)$, $(B_i \to A_4)(i \neq 4)$ and $(B_i \to A_6)$ $(i \neq 5)$ are empty and similarly as above we select the maximum θ(i, j) among $\theta(1,2)$, $\theta(2,1)$, $\theta(3,3)$ and $\theta(6,5)$. Then

$$\theta(1,2) = (B_1 \to A_3) + (B_2 \to A_2) = A_3B_{13} + A_2B_{31} = 2,$$

$$\theta(2,1) = (B_3 \to A_1) + (B_2 \to A_2) = A_2B_{23} + A_1B_{32} = 0,$$

$$\theta(3,3) = (B_3 \to A_2) + (B_2 \to A_3) = A_3B_{23} + A_2B_{32} = 0,$$

$$\theta(6,5) = (B_6 \to A_3) + (B_3 \to A_5)$$

$$=A_5B_{56}+A_4B_{45}+A_3B_{64}+A_5B_{35}+A_4B_{54}+A_3B_{43}=50$$

and $\theta(6,5)=50$ is maximum. Hence $B_6=B(3,4)$ corresponds to $A_5=A'(1,3)$ and $[\overline{B_6\to A_5}]$ and $[B_6\to A_5]$ are the branches of $[B_5\to A_6]$.

Now the lower bound ot $[\overline{B_6\to A_5}]$ is $401+50=451$. On the other hand the lower bound of $[B_6\to A_5]$ is computed as follows; $B_2=B(1,2)$ must correspond to $A_1=A(2,4)$ since $B_6=B(3,4)$ corresponds to $A_5=A'(1,3)$ but $B_5=B(1,4)$ corresponds to $A_6=A'(1,2)$. Thus we have the following assignment $\begin{pmatrix}1 & 2 & 3 & 4\\ a_4 & a_1 & a_3 & a_2\end{pmatrix}$ and $B_1=B(1,3)$ corresponds to $A_3=A'(2,3)$. Hence the lower bound is $401+(B_1\to A_3)=401+2=403$ and, since 403 is smaller than 451, $\begin{pmatrix}1 & 2 & 3 & 4\\ a_4 & a_1 & a_3 & a_2\end{pmatrix}$ is the optimal assignment.

References

1) J. W. Gavett and Norman V. Plyter; "The Optimal Assignment of Facilities to Locations by Branch and Bound", Opns. Res. 14, No. 2, 210—232 (1966).

2) J. D. C. Little, K. G. Murty, D. W. Sweeneyand C. Karel, "An Algorithm for the Traveling Salesman Problem", Opns. Res. 11, 972—989 (1963).