Approximate Solution of Nonlinear Oscillatory Circuits (II)

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In the preceding paper a new method of analyzing nonlinear periodic oscillations was proposed. In this article a new conception, which is named as the extended harmonic approximation of nonlinear oscillatory circuits, is presented. Method of obtaining transient solutions by the conception is given and various numerical examples are shown. The new conception has merits that a consistent linearization method is given for both steady state and transient state domains and transient solutions are obtained simply.

§ 1. Introduction

The harmonic approximation of nonlinear oscillatory circuits defined in the preceding paper[1] proposed a simple method to obtain steady solutions. The present paper deals with a new conception, which is named as the extended harmonic approximation of nonlinear oscillatory circuits. The method of analysis of transient state according to the conception is mentioned. It is shown that the harmonic approximation is a special case of the extended harmonic approximation and the analysis of transient state by the method of harmonic approximation is explained by various examples. The new conception has remarkable merits that a consistent linearization is possible for both steady state and transient state domains and transient solutions can be obtained simply.

An unification and extension of miscellaneous linearization methods by the present conception will be treated in a following paper.

§ 2. Definition of the extended harmonic approximation

For the time being, it is assumed that always at the end of transient state a periodic oscillation takes place, and the oscillation is represented in the following form:

$$v(t)=v_0+\sum_{n=1}^{n}v_n,\quad v_0=v_0(t),$$
$$v_n=v_{nm}(t)\sin(n\omega t-\theta_n(t)). \qquad (1)$$

We assume that $v_{nm}(t)$, $\theta_n(t)$ and $v_0(t)$ are slowly varying functions of time compared with the periodic oscillation $2\pi/\omega$. Moreover, when $t\to\infty$, it is assumed that the oscillation becomes periodic, namely

$$v=v_{0\infty}+\sum_{n=1}^{n}v_{n\infty},\quad v_{n\infty}=v_{nm\infty}\sin(n\omega t-\theta_{n\infty}),$$

where $v_{0\infty}$, $v_{nm\infty}$, $\theta_{n\infty}$ are independent of time. The transient state is represented in Eq. (1) as the amplitude modulation $v_{nm}(t)$ and phase modulation $\theta_n(t)$ of the periodic oscillation. Then, putting v into the nonlinear term $f(v)$, we expand $f(v)$ into Fourier type series[2]. From the series we take up the same frequency terms as v and make a summation f_h of these terms. That is,

$$f_h=A_0+\sum_{n=1}^{n}A_n\sin(n\omega t-\theta_n)$$
$$+\sum_{n=1}^{n}B_n\cos(n\omega t-\theta_n), \qquad (2)$$

where

$$A_0=\frac{1}{2\pi}\int_0^{2\pi}fd(\omega t),$$
$$A_n=\frac{1}{\pi}\int_0^{2\pi}f\sin(n\omega t-\theta_n)d(\omega t),$$
$$B_n=\frac{1}{\pi}\int_0^{2\pi}f\cos(n\omega t-\theta_n)d(\omega t),$$
$$f=f(v)=f\{v_0+\sum_{n=1}^{n}v_{nm}\sin(n\omega t-\theta_n)\}.$$

Here, assuming that time t in $v_0(t)$, $v_{nm}(t)$, $\theta_n(t)$ is a parameter, we calculate Fourier coefficients. A_n, B_n are dependent of time because of t in the amplitude $v_{nm}(t)$ and phase $\theta_n(t)$. Therefore, referring to the preceding paper, we define it as the extended (0, 1, $\cdots$n) harmonic approximation of nonlinear element.

Next, let us represent Eqs. (1)~(2) in vector form. First we represent in polar coordinates as follows:

$$V(t)=V_0+\sum_{n=1}^{n} V_n,\quad V_0=v_0,\quad V_n=v_{nm}\varepsilon^{j(n\omega t-\theta_n)},$$

$$F_h=A_0+\sum_{n=1}^{n} A_n\varepsilon^{j(n\omega t-\theta_n)}+\sum_{n=1}^{n} B_n j\varepsilon^{j(n\omega t-\theta_n)}$$

$$=\frac{A_0}{v_0}V_0+\sum_{n=1}^{n}\frac{A_n}{v_{nm}}V_n+j\sum_{n=1}^{n}\frac{B_n}{v_{nm}}V_n,$$

$\therefore$

$$F_h=\alpha_0 V_0+\sum_{n=1}^{n}(\alpha_n+j\beta_n)V_n, \tag{3}$$

where

$$\alpha_0=\frac{1}{v_0 2\pi}\int_0^{2\pi} f\,d(\omega t),$$

$$\alpha_n=\frac{1}{v_m\pi}\int_0^{2\pi} f\varepsilon^{j(n\omega t-\theta_n)}d(\omega t),$$

$$\beta_n=\frac{1}{v_m\pi}\int_0^{2\pi} fj\,\varepsilon^{j(n\omega t-\theta_n)}d(\omega t).$$

The instantaneous value v is the real or imaginary part of vector V. Accordingly α_n, β_n become the real or imaginary part of vector $\boldsymbol{\alpha}_n$, $\boldsymbol{\beta}_n$. We calculate the differential coefficients of V_n because of their necessity in later sections. That is,

$$V_n=v_{nm}\varepsilon^{j(n\omega t-\theta_n)},$$

$$\dot{V}_n=\{\dot{v}_{nm}-j(v_{nm}\dot{\theta}_n-n\omega v_{nm})\}\varepsilon^{j(n\omega t-\theta_n)},$$

$$\ddot{V}_n=\{\ddot{v}_{nm}-v_{nm}\dot{\theta}_n^2-j(\dot{v}_{nm}\dot{\theta}_n+v_{nm}\ddot{\theta}_n+\dot{\theta}_n\dot{v}_{nm})$$
$$+2n\omega v_{nm}\dot{\theta}_n-n^2\omega^2 v_{nm}$$
$$+j2n\omega\dot{v}_{nm}\}\varepsilon^{j(n\omega t-\theta_n)},$$

....

Assuming that the amplitude and phase vary slowly with time, we neglect small differential terms. Thus we obtain the following approximations:

$$\left.\begin{aligned}&V_n=v_{nm}\varepsilon^{j(n\omega t-\theta_n)},\\&\dot{V}_n\doteqdot jn\omega v_{nm}\varepsilon^{j(n\omega t-\theta_n)},\\&\ddot{V}_n\doteqdot(2n\omega v_{nm}\dot{\theta}_n-n^2\omega^2 v_{nm}\\&\qquad+j2n\omega\dot{v}_{nm})\varepsilon^{j(n\omega t-\theta_n)},\\&\cdots\cdots\end{aligned}\right\} \tag{4}$$

Especially, where nonlinear element is of a form $f(v,\ \dot{v})$, the differentiation of Eq. (1) becomes as

$$\dot{v}=\sum_{n=1}^{n} n\omega v_{nm}(t)\cos(n\omega t-\theta_n(t)).$$

Therefore

$$f(v,\dot{v})=f\{v_0+\sum_{n=1}^{n}v_{nm}\sin(n\omega t-\theta_n),$$
$$\sum_{n=1}^{n}n\omega v_{nm}\cos(n\omega t-\theta_n)\}.$$

Putting $f(v,\ \dot{v})$ into the relation of Fourier coefficients, we obtain A_0, A_n, B_n. The above differentiation $\dot{v}$ means that the amplitude and phase are slowly varying functions of time.

Next, in rectangular coordinates, we obtain as follows:

$$x_n=v_{nm}\sin\theta_n,\quad y_n=v_{nm}\cos\theta_n,$$

$$F_h=\alpha_0 V_0+\sum_{n=1}^{n}(\alpha_n+j\beta_n)(y_n-jx_n)\varepsilon^{jn\omega t},$$

$$\left.\begin{aligned}&V_n=(y_n-jx_n)\varepsilon^{jn\omega t},\\&\dot{V}_n=\{\dot{y}-j\dot{x}_n+n\omega(x_n+jy_n)\}\varepsilon^{jn\omega t},\\&\ddot{V}_n=\{\ddot{y}_n-j\ddot{x}_n+2n\omega(\dot{x}_n+j\dot{y}_n)\\&\qquad-n^2\omega^2(y_n-jx_n)\}\varepsilon^{jn\omega t},\\&\cdots\cdots\end{aligned}\right\} \tag{5}$$

When the amplitudes x_n, y_n also are slowly-varying functions of t, we can approximate as

$$\left.\begin{aligned}&V_n=(y_n-jx_n)\varepsilon^{jn\omega t},\\&\dot{V}_n\doteqdot n\omega(x_n+jy_n)\varepsilon^{jn\omega t},\\&\ddot{V}_n\doteqdot\{2n\omega(\dot{x}_n+j\dot{y}_n)\\&\qquad-n^2\omega^2(y_n-jx_n)\}\varepsilon^{jn\omega t},\\&\cdots\cdots\end{aligned}\right\} \tag{6}$$

Eqs. (3), (5) are the extended (0, 1, ···, n) harmonic approximation represented with vector notation of nonlinear element. Taking the real or imaginary part of the above vector equations, we can obtain the relations of instantaneous value. The physical system having the extended (0, 1, ···, n) harmonic approximation instead of nonlinear element in the original nonlinear system is linear conditionally and defined as the "extended (0, 1, ···, n) harmonic approximation" (or extended (0, 1, ···, n) harmonically approximated system) of the original system. The case where n=1, we name it simply the "extended harmonic approximation." When the phenomenon approaches to the steady state, then

$$\dot{v}_{nm}=\ddot{v}_{nm}=\cdots=0,\quad \dot{\theta}_n=\ddot{\theta}_n=\cdots=0,$$

$$\dot{x}=\ddot{x}_n=\cdots=0,\qquad \dot{y}_n=\ddot{y}_n=\cdots=0.$$

Therefore the extended (0, 1, ···. n) harmonic approximation coincide with the (0, 1, ···, n) harmonic approximation.

Example 1

We take Duffing's equation:

$$m\ddot{v}+c\dot{v}+f(v)=p\sin\omega t,\quad f(v)=kv+bv^3.$$

We assume the solution as

$$v=v_m(t)\sin(\omega t-\theta(t)).$$

Namely, it is represented as the imaginary part of vector V. Performing the transformation $v\to V$, $\dot{v}\to\dot{V}$, $\ddot{v}\to\ddot{V}$, $f\to F_h=\alpha V$, $p\sin\omega t\to p\varepsilon^{j\omega t}$ in the original equation, we obtain the linearization, namely the extended harmonic approximation, as follows:

$$m\ddot{V}+c\dot{V}+\alpha V=p\varepsilon^{j\omega t},$$

where

$$\alpha=\frac{1}{v_m\pi}\int_0^{2\pi}f\{v_m\sin(\omega t-\theta)\}\sin(\omega t-\theta)d(\omega t)$$
$$=k+\frac{3}{4}b\{v_m(t)\}^2.$$

Taking the imaginary part of the equation the instantaneous representation becomes as

$$m\ddot{v}+c\dot{v}+\alpha v=p\sin\omega t.$$

Since α depends on time because of the time function $v_m(t)$, those extended harmonic approximations are linear conditionally and governed by linear differential equations with variable coefficients.

Example 2

We take the unsymmetrical nonlinear system:

$$m\ddot{v}+c\dot{v}+f(v)=p\sin\omega t,\quad f(v)=kv+bv^2.$$

We assume the solution as $v=v_0+v_1=v_0(t)+v_m(t)\sin(\omega t-\theta(t))$.

The linearization, in vector notation, of nonlinear term becomes

$$V_0=v_0,\quad V_1=v_m\varepsilon^{j(\omega t-\theta)},\quad F_h=\alpha_0V_0+\alpha_1V_1,$$
$$\alpha_0=k+bv_0+\frac{bv_m^2}{2v_0},\qquad \alpha_1=k+2bv_0,$$

by reference to the preceding paper. Performing the transformation $v\to V_0+V_1$, $\dot{v}\to\dot{V}_0+\dot{V}_1$, $\ddot{v}\to\ddot{V}_0+\ddot{V}_1$, $f\to F_h$, $p\sin\omega t\to p\varepsilon^{j\omega t}$ in the original equation, we obtain the extended (0, 1) harmonic approximation as follows:

$$m(\ddot{V}_0+\ddot{V}_1)+c(\dot{V}_0+\dot{V}_1)+\alpha_0V_0+\alpha_1V_1=p\varepsilon^{j\omega t}.$$

According to the principle of harmonic balance, we obtain the following equations:

$$\begin{cases}m\ddot{V}_0+c\dot{V}_0+\alpha_0V_0=0,\\ m\ddot{V}_1+c\dot{V}_1+\alpha_1V_1=p\varepsilon^{j\omega t}.\end{cases}$$

The instantanous representations become

$$\begin{cases}m\ddot{v}_0+c\dot{v}_0+\alpha_0v_0=0,\\ m\ddot{v}_1+c\dot{v}_1+\alpha_1v_1=p\sin\omega t.\end{cases}$$

Example 3

We take a subharmonic oscillation represented by the following equations:

$$m\ddot{v}+c\dot{v}+f(v)=p\cos 3t,\quad f(v)=kv+bv^3.$$

We assume the solution as

$$v=v_1+v_3,\quad v_1=v_{1m}(t)\cos(t-\theta_1(t)),$$
$$v_3=v_{3m}(t)\cos(3t-\theta_3(t)).$$

The linearizations in vector notation become

$$V_1=v_{1m}\varepsilon^{j(t-\theta)},\quad V_3=v_{3m}\varepsilon^{j(3t-\theta_3)},$$
$$F_h=\alpha_1V_1+j\beta_1V_1+\alpha_3V_3+j\beta_3V_3,$$
$$\therefore\begin{cases}m\ddot{V}_1+c\dot{V}_1+(\alpha_1+j\beta_1)V_1=0,\\ m\ddot{V}_3+c\dot{V}_3+(\alpha_3+j\beta_3)V_3=p\varepsilon^{j3t}.\end{cases}$$

The last equations represent the extended (1, 3) harmonic approximation. The forms of α, β are similar to those of the preceding paper. However, the amplitude and phase of α, β in this case must be replaced with time functions $v_{1m}(t)$, $v_{3m}(t)$, $\theta_1(t)$, $\theta_3(t)$.

Example 4

We take a system containing hysteresis element:

$$m\ddot{v}+c\dot{v}+f(v)=p\sin\omega t.$$

We assume the solution as $v=v_m(t)\sin(\omega t-\theta(t))$. The linearizations in vector notation become

$$V=v_m\varepsilon^{j(\omega t-\theta)},\qquad F_h=(\alpha+j\beta)V,$$
$$f=f\{v_m\sin(\omega t-\theta)\},$$
$$\alpha=\frac{1}{v_m\pi}\int_0^{2\pi}f\sin(\omega t-\theta)d(\omega t),$$
$$\beta=\frac{1}{v_m\pi}\int_0^{2\pi}f\cos(\omega t-\theta)d(\omega t),$$
$$\therefore\quad m\ddot{V}+c\dot{V}+(\alpha+j\beta)V=p\varepsilon^{j\omega t}.$$

The last equation represents the extended harmonic approximation.

§ 3. Method of solution of the extended harmonically approximated system

The solution of the extended harmonically approximated system is found adequately in the topological space. First we assume the solution in vector notation as $V=V_0+\sum_{k=1}^{n}V_n$ and obtain the derivatives:

$$\dot{V}=\dot{V}_0+\sum_{n=1}^{n}\dot{V}_n,\quad \ddot{V}=\ddot{V}_0+\sum_{n=1}^{n}\ddot{V}_n,\quad \text{etc.}$$

We substitute those derivatives into the linear differential equation with variable coefficients, which represent the extended (0, 1, ···, n) harmonically approximated system, and at the same time we refer Eq. (4) or (5) for the differentiations of V_n. Then, the real and imaginary part of both sides of the equation are respectively equated. Finally, simultaneous nonlinear differential equations for the amplitude and phase or for amplitudes x_n, y_n are obtained. The extended (0, 1, ⋯, n) harmonic approximation of original system is determined from the last equations.

Example

$$m\ddot{v}+c\dot{v}+f(v)=p\sin\omega t,\quad f(v)=kv+bv^3.$$

We assume the solution as $v=v_m\sin(\omega t-\theta)$. The linearization becomes as follows:

$$m\ddot{V}+c\dot{V}+\alpha V=p\varepsilon^{j\omega t}.$$

We assume a slow time variation of the amplitude and phase and substitute the Eq. (4) (n=1, in this case) into this equation. Equating respectively the real and imaginary part of both sides of the equation, we obtain the extended harmonic approximation as

$$\dot{v}_m=\frac{p\sin\theta-c\omega v_m}{2m\omega},\qquad \alpha=k+\frac{3}{4}bv_m^2,$$

$$\dot{\theta}=\frac{p\cos\theta+m\omega^2v_m-\alpha v_m}{2m\omega v_m}.$$

Those equations determine time variation of amplitude and phase. If Eq. (6) is used, the following equations are obtained:

$$\begin{cases}\dot{x}=\dfrac{1}{2m\omega}(-c\omega x+\alpha y-mc^2y),\\ \dot{y}=\dfrac{1}{2m\omega}(p-\alpha x+m\omega^2x-c\omega y),\\ \alpha=k+\dfrac{3}{4}b(x^2+y^2).\end{cases}$$

The solutions of these equations give solution curves of the extended harmonic approximation in the phase plane. The extended harmonically approximated solutions are constructed as follows:

$$v=v_m(t)\sin(\omega t-\theta(t)),$$

$$v=x(t)\sin\omega t-y(t)\cos\omega t.$$

§ 4. The first order approximation of linearization parameter α, β

The extended harmonically approximated system is represented by linear differential equations with variable coefficients. Furthermore, the equations determining the amplitude and phase become simultaneous nonlinear differential equations. Generally, the solutions are difficult to solve in either case. The difficulty is due to the fact that α, β depend on time. Therefore, the first approximation of α, β, not containing time, will be calculated in the phase space. The representative point $x_n(t)$, $y_n(t)$ in the phase space moves, with the lapse of time, on the solution curve and gets ultimately to a stable singular point. To discuss oscillation near a stable singular point, α, β can be expanded in Taylor's series about this point and usually it is sufficient to retain only the first term. That is, α, β at the stable singular point may be used in this case. Then, the system having the above α, β is nothing but the harmonic approximation mentioned in the preceding paper. In order that the transient solution in the harmonic approximation becomes the approximation of exact solution, it is required that whole solution curves of oscillation, with the lapse of time, retain near the stable singular point. The application limit of transient solution by the harmonic approximation will be explained in detail in the next paragraph.

The linearizations stated in this study are illustrated as follows:

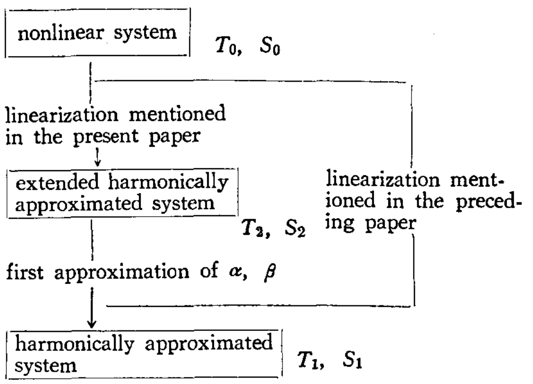

Fig. 1 Table of the linearizations.

Where T and S show transient and periodic solutions respectively, and 0, 1 and 2 show the exact solutions, the harmonic approximations and the extended harmonic approximations respectively. Generally,

$$S_1=S_2\fallingdotseq S_0,\quad T_2\fallingdotseq T_0,\quad T_1\neq T_2,$$

but when some conditions mentioned in the next paragraph are satisfied, then $T_1\fallingdotseq T_2$.

§ 5. Transient solution of the harmonically approximated system

Example 1

We take a system with symmetrical nonlinear element. For instance,

$$m\ddot{v}+c\dot{v}+f(v)=p\sin\omega t,\quad f(v)=kv+bv^3. \quad (7)$$

We assume the solution as

$$v=v_m\cos(\omega t-\theta)=x\sin\omega t-y\cos\omega t,$$

$$x=v_m\sin\theta,\quad y=v_m\cos\theta.$$

As mentioned above, the first approximation of the linearization parameters α, β is valid only near a stable singular point or a periodic oscillation $v_{m\infty}$, θ_∞. We shall assume now that the approximate periodic solution of the original nonlinear system has been obtained with an analogue computer as was mentioned in preceding paper. When the trajectory retain near a stable singular point, the transient oscillatory part of the solution curve, namely the forerunner state moving to the steady state, becomes a sufficient approximation of exact solution. Accordingly we do not necessitate special operations for the transient solution. If we calculate the approximate periodic solution by an analogue computer, as mentioned in the preceding paper, we also obtain simultaneously the approximate transient solution.

Using Eq. (5), the above system is linearized as

$$m\ddot{V}+c\dot{V}+(\alpha+j\beta)(y-jx)\varepsilon^{j\omega t}=-jp\varepsilon^{j\omega t},$$

where

$$\alpha=k+\frac{3}{4}bv_m^2,\quad \beta=0.$$

We assume that x, y are slowly varying functions of t. Substituting Eq. (6) into the above equation, and equating the real and imaginary part of both sides of the equation respectively, we obtain the equations determining time variation of x, y as follows:

$$\left.\begin{aligned}\dot{x}&=\frac{1}{2m\omega}(-c\omega x-\alpha y+m\omega^2 y),\\ \dot{y}&=\frac{1}{2m\omega}(-p+\alpha x-m\omega^2 x-c\omega y),\\ \alpha&=k+\frac{3}{4}b(x^2+y^2).\end{aligned}\right\}\quad (8)$$

Eq. (8) gives the trajectory of the extended harmonically approximated system. When the phenomenon is restricted near a stable singular point, the time variations of x, y will be given as

$$\left.\begin{aligned}\dot{x}&=\frac{1}{2m\omega}(-c\omega x-\alpha_\infty y+m\omega^2 y),\\ \dot{y}&=\frac{1}{2m\omega}(-p+\alpha_\infty x-m\omega^2 x-c\omega y),\\ \alpha_\infty&=k+\frac{3}{4}b(x_\infty^2+y_\infty^2).\end{aligned}\right\}\quad (9)$$

Eq. (9) gives solution curves of the harmonically approximated system. The equation of oscillation for the harmonic approximation or Eq. (9) becomes as

$$m\ddot{v}+c\dot{v}+\alpha_\infty v=p\sin\omega t. \quad (10)$$

Eqs. (7) and (8) can be solved by an analogue computer with multipliers and Eqs. (9), (10) by an analogue computer with only linear elements. Eqs. (9), (10) give the approximation of Eqs. (8), (7) respectively. Now to show an numerical example, we make the same assumption as mentioned in paper 1, paragraph 3, example 1. Draw-

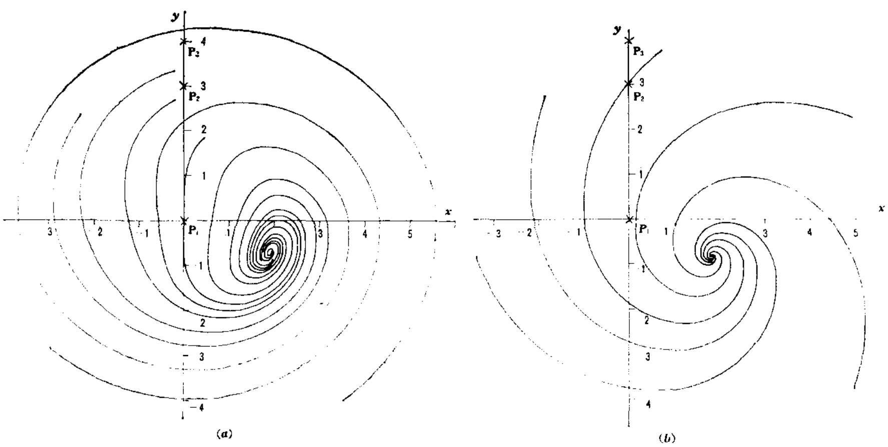

Fig. 2 Solution curves. (u=1)
(a) Solution curve of Eq. (8). b) Solution curve of Eq. (9).

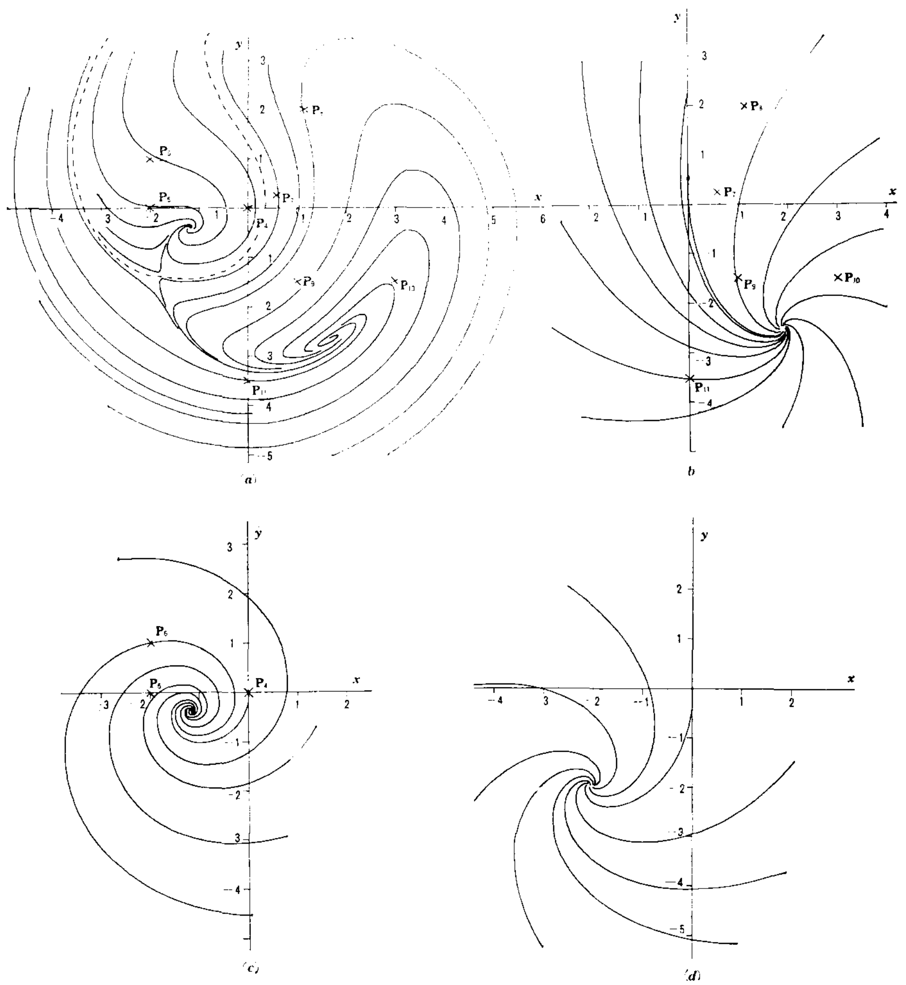

Fig. 3 Solution curves. (u=1.35)
(a) Solution curve of Eq. (8). (b) Solution curve of Eq. (9).
(c) Solution curve of Eq. (9). (d) Solution curve of Eq. (9).

ing solution curves at $\omega=1$ and $\omega=1.35$, we obtain Figs. 2~3. Where scale factors are $\alpha_v=X/x=Y/y=1$, $\alpha_t=T/t=\sqrt{k/m}$ and $u=\omega/\alpha_t$. Figs. 2 (a), 3(a) are obtained from Eq (8) and Figs. 2(b), 3(b), 3(c), 3(d) from Eq. (9).

Trajectories near a stable singular point, namely Figs 2(a), 2(b) or Figs. 3(a), 3(c), are considerably similar. Position of stable singular points in Figs. 3(a), 3 (b) is considerably similar, but trajectories are not. Fig. 3(d) shows trajectory of the harmonically approximated system, corresponding to an unstable singular point in Fig. 3(a). Similarity of both trajectories means that transient solutions of x, y and moreover transient oscillations of Eqs. (7), (10) are considerably similar respectively. However, the curve of oscillation is not the trajectory itself, but has the representation $v=x\sin\omega t+y\cos\omega t$ including $\sin\omega t$ and $\cos\omega t$. Accordingly, velocity of the representative point moving on a trajectory is also related. Similarity of both curves of oscillation by Eq. (7) (10) is not completely explained from similarity of both trajectories. But, trajectory gives us an effective clue of analysis.

Taking into account the initial conditions corresponding to points p_1~p_{11} in Figs. 2 and 3, we solve Eqs. (7), (10) by a computer and discuss the results. The method of operation was already mentioned in the preceding paper. We obtain the results as Figs. 4~14. In these figures, (a) show exact solution by a computer with multipliers and (b) show solutions of Eq.

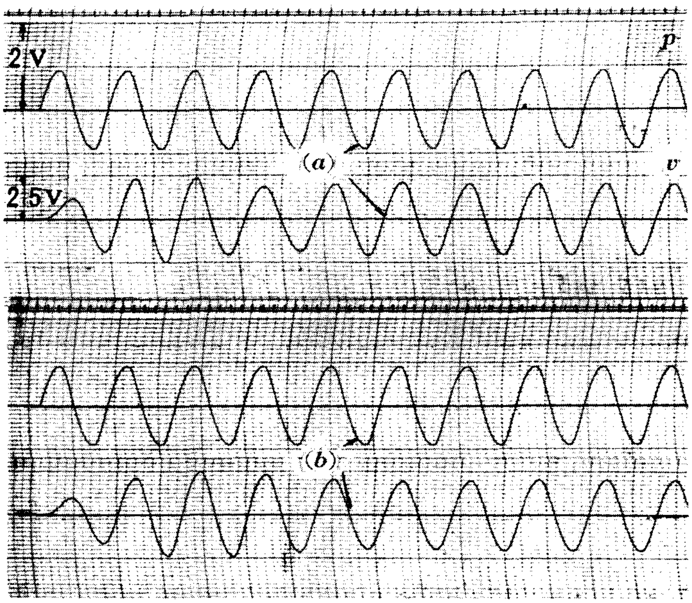

Fig. 4 Comparison between the exact solution (a) and the harmonic approximation (b). (p_1, u=1)

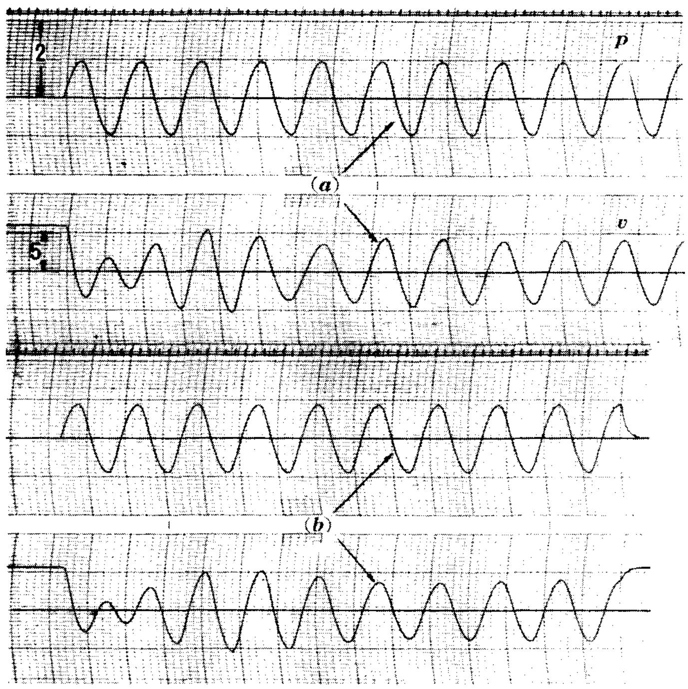

Fig. 5 Comparison between the exact solution (a) and the harmonic approximation (b). (p_2, u=1)

(10) by a computer with only linear elements. Both oscillation curves on Fig. 4 show a considerable similarity. When the distance between an initial point and a stable singular point separates gradually, oscillation curves do not change equally gradually. Especially, if the initial point is far away from a stable singular point as p_3, the transient part of oscillation illustrated by Fig. 6 differs considerably. Both trajectories which pass through the initial point p_3 apparently differ as Figs. 2(a), 2(b). The point p_4 is situated near a separatrix of trajectory as shown in Fig. 3. A representative point on a separatrix moves with the lapse of time and gets ultimately to an unstable singular point. The velocity of the movement of the representative point relaxes near an unstable singular point and becomes zero at this point theoretically. In fact, the representative point is forced to move by slight disturbances and gets ultimately to a stable singular point. Accordingly, when the initial point is accidentally in the proximity of the separatrix, the transient solution of the original nonlinear system continue very long accrding to the degree of approximation to the separatrix. Eq. (10), namely the linearized system, has not this separatrix. Terefore, it does not occur that the transient state continue very long according to inital condition. From this standpoint it will be explained the difference of (a) and (b) in Fig. 7. Both oscillation curves corresponding to initial points p_5, p_6, namely Figs. 8 and 9, are considerably similar. When Fig. 10, corresponding to initial point p_7, is observed in detail, we know that the transient solution (a) continues long, because of its approximation to separatrix. Fig. 11, corresponding to initial point p_8, shows some difference because of its large separation from a stable singular point. Both oscillation curves corresponding to initial points p_9, p_{10}, p_{11}, namely Figs. 12~14, show considerable similarity. But, those trajectories in Figs. 3(a), 3(b) do not

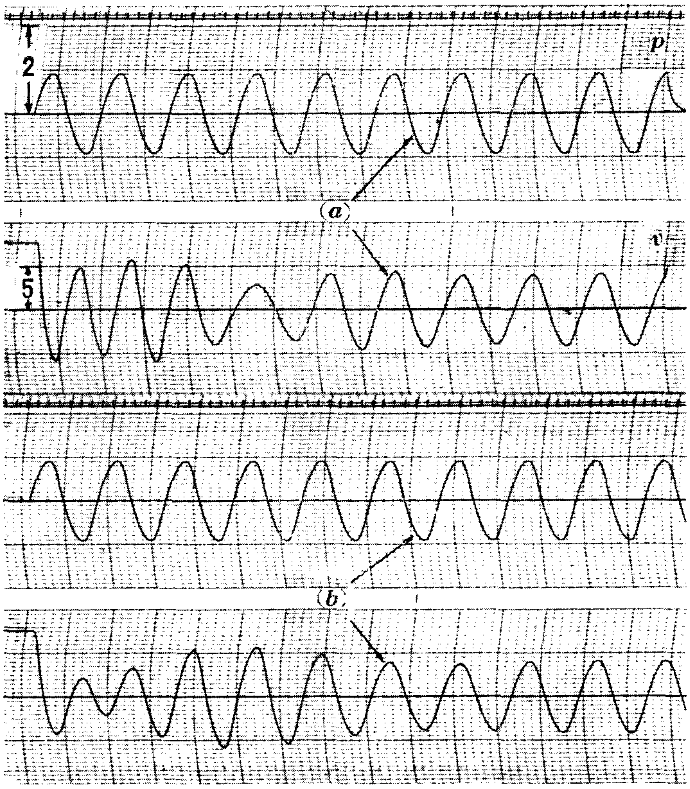

Fig. 6 Comparison between the exact solution (a) and the harmonic approximation (b). (p_3, u=1)

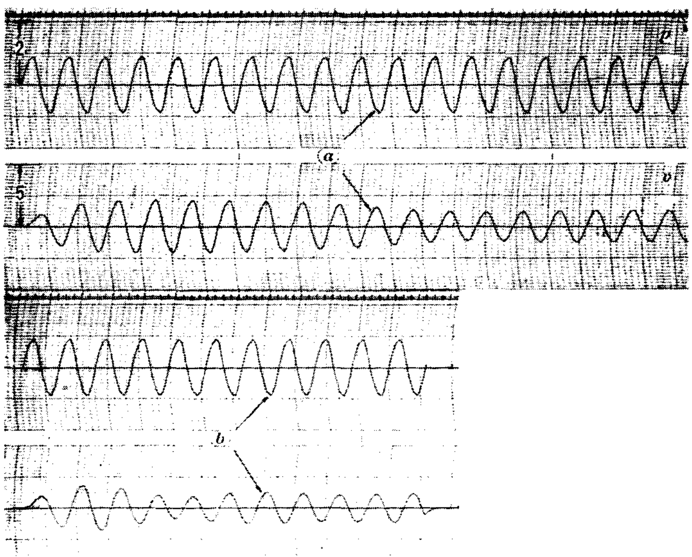

Fig. 7 Comparison between the exact solution (a) and the harmonic approximation (b). (p_4, u=1.35)

simulate quite well.

The application limit of the harmonic approximation in transient state has been roughly explained by the above statement. In other words, it is necessary that in the total lapse of time the trajectory do not depart largely from a stable singular point. This requirement is closely related to that the nonlinearity is not very great. Furthermore, it is also necessary that the initial value is not situated too near the separatrix.

Example 2

We take a system with unsymmetrical nonlinear element. For instance,

$$m\ddot{v}+c\dot{v}+f(v)=p\sin\omega t,$$
$$f(v)=kv+bv^2.$$

Operations of culculation and results of these equations were already mentioned in report 1, paragraph 3, example 2. As another example, we take a system having a constant forcing term as follows:

$$m\ddot{v}+c\dot{v}+f(v)=p\sin\omega t+q,$$
$$f(v)=kv+bv^3.$$

Assuming the solution as $v=v_0+v_1$, $v_1=v_m\sin(\omega t-\theta)$, we obtain the (0, 1) harmonic approximation as follows:

$$f_h=\alpha_{0\infty}v_0+\alpha_{1\infty}v_1,$$
$$\alpha_{0\infty}=bv_0^2+\frac{3}{2}bv_m^2+k,$$
$$\alpha_{1\infty}=3bv_0^2+\frac{3}{4}bv_m^2+k,$$

$\therefore$

$$m\ddot{v}+c\dot{v}+\alpha_{1\infty}v_1=p\sin\omega t,$$
$$\alpha_{0\infty}v_0=q,$$

or $m\ddot{v}+c\dot{v}+\alpha_{1\infty}v$

$$=p\sin\omega t+\alpha_{1\infty}v_0.$$

The last equation is similar to the equation (4) in the preceding paper and accordingly the operation is also analogous. To show an numerical example, we assume as follows:

$$C/\sqrt{km}=0.17,\quad b/k=0.133,$$
$$p/k=0.8,\quad q/k=2,$$
$$u=\omega/\sqrt{k/m},\quad \alpha_t=\sqrt{k/m}.$$

Figs. 15~16 show some of the results. In these figures, (a) are the exact solution with multipliers and (b) are the harmonic approximation. From Figs 15~16 we know that the harmonic approximation is the approximation of the exact solution even at transient state.

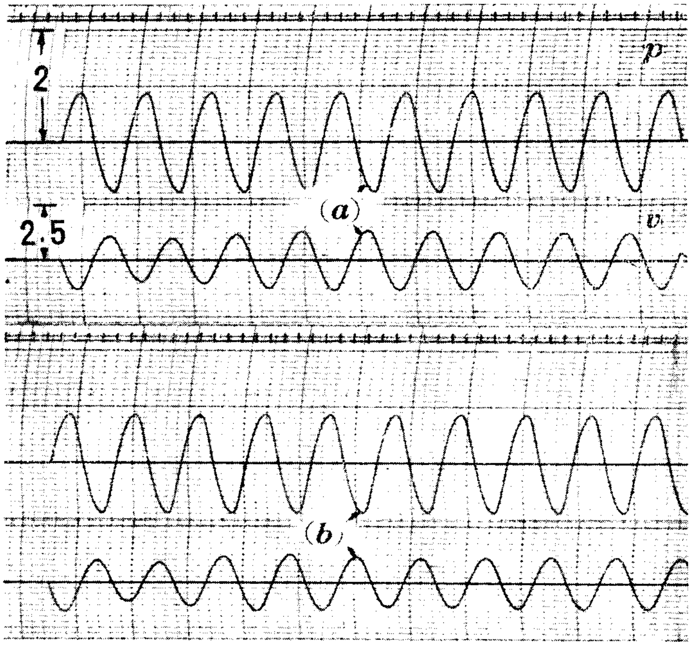

Fig. 8 Comparison between the exact solution (a) and the harmonic approximation (b). (p^5, u−1.35)

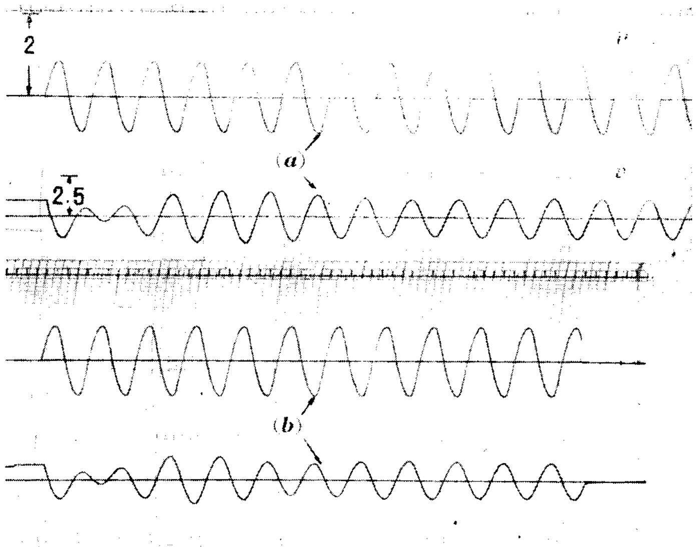

Fig. 9 Comparison between the exact solution (a) and the harmonic approximation (b). (p_6, u=1.35)

Example 3

We take the system containing hysteresis element. For instance,

$$N\dot{\phi}+Ri+\frac{1}{C}\int idt$$
$$=\sqrt{2}\,E\sin\omega t=e,$$
$$i=f(\phi),\quad R=1.12,$$
$$C=40.8\times10^{-6},\quad N=242,$$

and $\alpha_\varphi=10^4$, $\alpha_i=10$, $\alpha_t=200$ (scale factors). Operations and results of this system were already mentioned also in report 1, paragraph 3, example 3. Now two other results will be supplemented in Figs. 17~18.

In those figures, (a) give the experimental results with a synchroscope and (b) give the harmonic approximations and both solutions agree considerably well. Frequently, to avoid the complexity of hysteresis of the core, only the effect of saturation is taken into account, and the nonlinearity is approximated by the curve of third order term of ϕ and i. If we assume that the nonlinearity is roughly approximated by the neutral line, as represented by the following equation, of hysteresis loop of the core used in the example:

$$i=a\phi+b\phi^3,\quad a=500,\quad b=10^9,$$

and again calculate the case having the same circuit conditions as Fig. 12 in report 1, paragraph 3, example 3. As shown in Fig, 19, we obtain a result that is apparently an almost periodic oscillation Of course, the result does not agree with the exact solution. We must be careful of the fact that, in some cases, neglec-

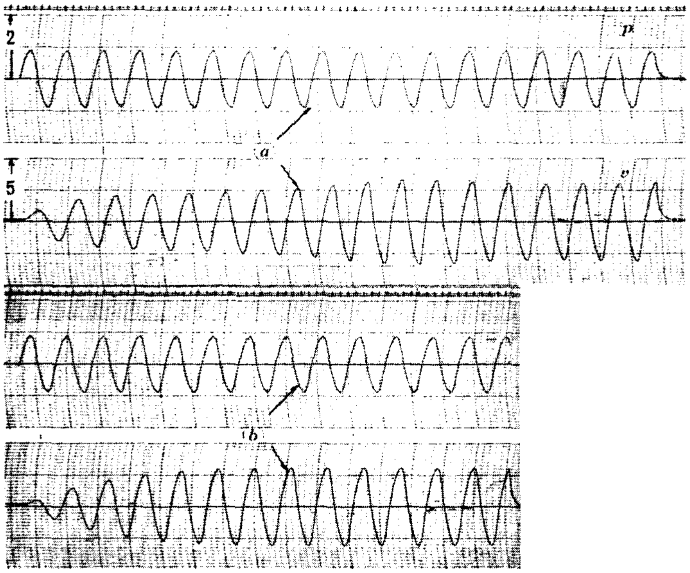

Fig. 10 Comparison between the exact solution (a) and the harmonic approximation (b). (p_7, u=1.35)

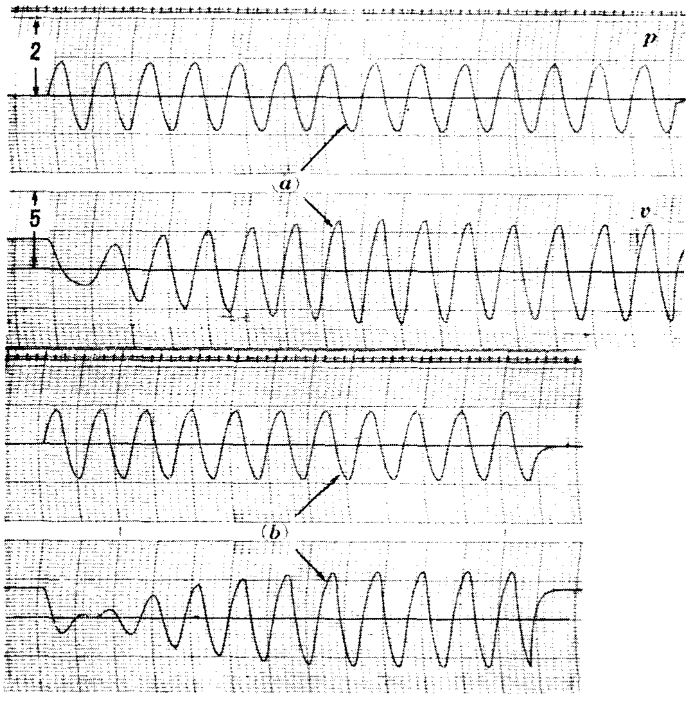

Fig. 11 Comparison between the exact solution (a) and the harmonic approximation (b). (p_8, u=1.35)

tion of hysteresis leads to a very false result. Once ϕ is obtained, the harmonic approximation of current $i_h = f_h$ can be simply obtained. That is, this current is nothing but i_1 in Fig. 10 of the preceding paper. Putting the outputs ϕ and i_h into X-Y recorder, we obtain the hysteresis loop as Fig. 20. A gradual growth of the loop is well understood. When the phenomenon approaches to the steady state, the loop becomes a hysteresis loop of ellipsoidal shape. The actual current is obtained graphically using the static hysteresis loop and the harmonic approximation ϕ. The sign(0) in Fig. 21 shows the results of this graphical calculation and roughly agrees with the experimental solution of current obtained by a synchroscope.

Discussion

The following conditions are required in order that the harmonic approximation in transient state approximate the exact solution.

(1) The nonlinearity is not very great.

(2) The initial value is not situated too near the separatrix.

We take the following equation which has not so marked nonlinearity:

$$\ddot{v} + \omega_0^2 v = \varepsilon f(v, \dot{v}, \omega t)$$

$$(0 < \varepsilon \ll 1).$$

The steady harmonic oscillation of this equation is named the pseudoharmonic or quasi-harmonic oscillation. If not very large initial value is given, the harmonic approximation can be applied to the above system and simply gives the approximate transient solution. But, in general, the harmonic approximation might not be applied to the automatic control system having very large nonlinearity. Generally, it will be dangerous to apply the harmonic approximation to the subharmonic oscillation having many separatrices in phase plane, except that the solution curves of phase plane of the extended harmonically approximated system are prepared at hand.

§ 6. Applications to autonomous systems

We take a damped oscillation as the example of autonomous system in transient state. The foundations of linearization in this paper are based on the periodic solution as mentioned at the beginning of the paragraph 2. Therefore, some contrivance is necessary for the application to the damped oscillation.

Example 1

$$\ddot{v}+c\dot{v}+v+bv^3=0. \tag{11}$$

Transforming the equation, we obtain as follows:

$$\ddot{v}+v=-c\dot{v}-bv^3\equiv-f(v,\dot{v}). \tag{12}$$

The generating solution of Eq. (12) is $v=v_m\cos(t-\theta)$. Then, we assume that the term $f(v, \dot{v})$ is a small perturbation. If the generating solution is adopted instead of the periodic solution, formally, the linearization of the present study can be applied in the same manner to the above system. Assuming that $f(v, \dot{v})$ is small and v_m, θ are slowly varying functions of t, we obtain the following linearization. For the extended harmonic approximation of nonlinear element we obtain as follows:

$$F_h=(\alpha+j\beta)V,$$

$$f=f\{v_m\cos(t-\theta), -v_m\sin(t-\theta)\},$$

$$\alpha=\frac{1}{v_m\pi}\int_0^{2\pi}f\cos(t-\theta)dt=\frac{3}{4}bv_m^2,$$

$$\beta=\frac{1}{v_m\pi}\int_0^{2\pi}f\sin(t-\theta)\,dt=c.$$

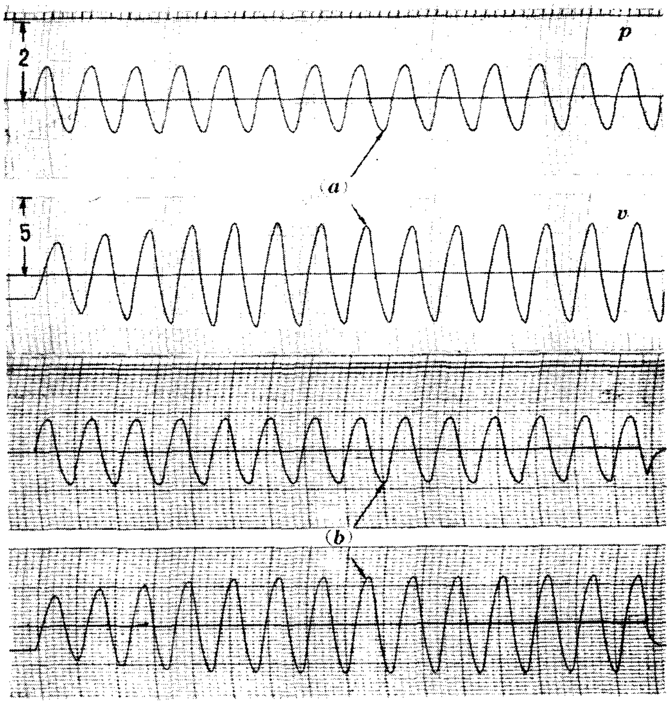

Fig. 12 Comparison between the exact solution (a) and the harmonic approximation (b). (p_9, u=1.35)

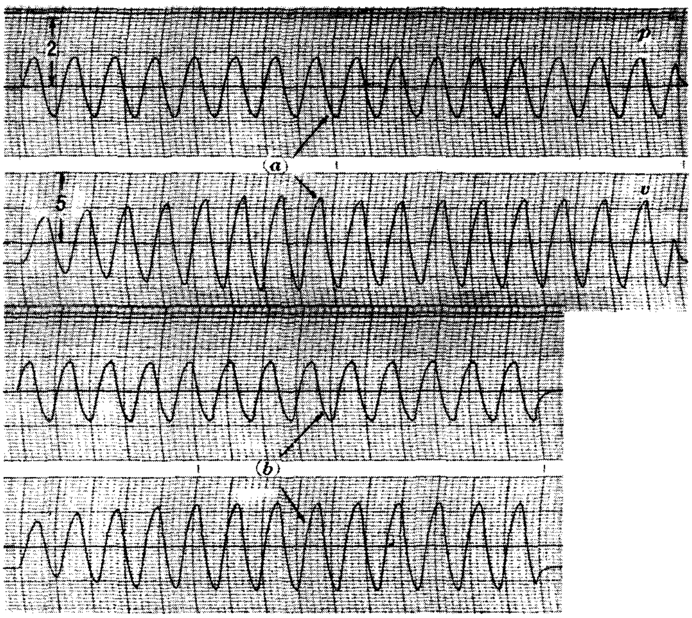

Fig. 13 Comparison between the exact solution (a) and the harmonic approximation (b). (p_{10}, u=1.35)

For the extended harmonic approximation of nonlinear system we obtain as follows:

$$\ddot{V}+V=-(\alpha+j\beta)V. \tag{13}$$

We calculate V, $\dot{V}$, $\ddot{V}$ by Eq. (6) and substitute into Eq. (13) and equate respectively the real and imaginary part of the equation. Then, we obtain the following equations representing solution curves of the extended harmonic approximation:

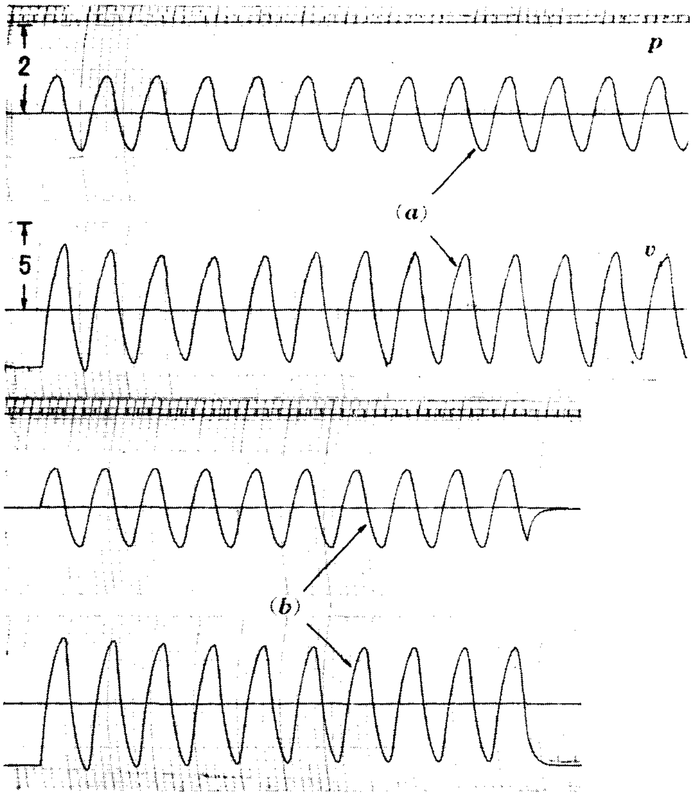

Fig. 14 Comparison between the exact solution (a) and the harmonic approximation (b). (p11, u=1.35)

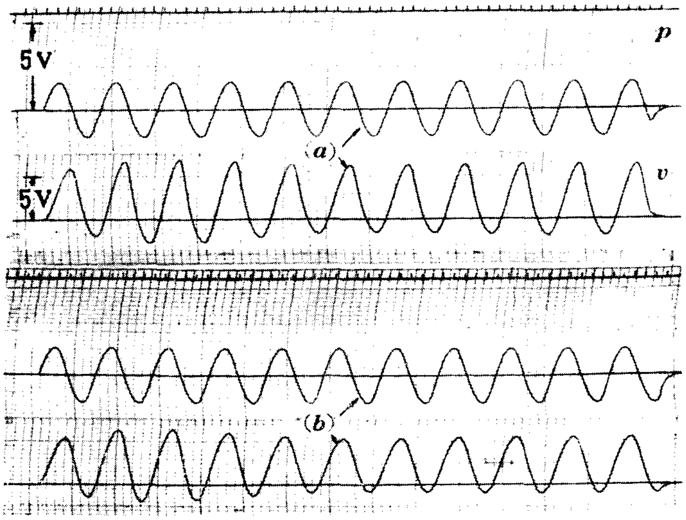

Fig. 15 Comparison the exact solution (a) and the (0, 1) harmonic approximation (b). (u=1.18)

$$\dot{x}=\frac{-\beta x-\alpha y}{2},\qquad \dot{y}=\frac{\alpha x-\beta y}{2},\qquad \alpha=\frac{3}{4}b(x^2+y^2),\qquad \beta=c. \tag{14}$$

From the solution of Eq. (14) the extended harmonic approximation is constructed as follows:

$$v=x\sin t+y\cos t.$$

Example 2

$$\ddot{v}+2\lambda\dot{v}+k^2v=\varepsilon f(v,\dot{v}),$$

$$(0<\varepsilon\ll 1).$$

When the system has a large damping term, and therefore a large coefficient of $\dot{v}$, considerable error is introduced by the above method. To avoid the error due to the large damping term, we take up a generating solution, which has a form of damped oscillation, as was adopted by Mr. E. P. Popov. Namely, as the generating solution the following form is assumed:

$$v=v_m\varepsilon^{-\lambda t}\sin(\omega t-\theta)$$

$$\equiv\rho\sin(\omega t-\theta),$$

where $\omega=\sqrt{k^2-\lambda^2}$ is the conditional frequency. We obtain the linearization in vector notation as follows:

$$\left.\begin{aligned}&V=\rho\varepsilon^{j(\omega t-\theta)},\\&F_h=(\alpha+j\beta)V,\\&\ddot{V}+2\lambda\dot{V}+k^2V\\&=\varepsilon(\alpha+j\beta)V,\end{aligned}\right\} \tag{15}$$

where

$$\alpha=\frac{1}{\rho\pi}\int_0^{2\pi}f\sin(\omega t-\theta)d(\omega t),$$

$$\beta=\frac{1}{\rho\pi}\int_0^{2\pi}f\cos(\omega t-\theta)d(\omega t),$$

$$f=f\{\rho\sin(\omega t-\theta),$$
$$\rho\omega\cos(\omega t-\theta)-\lambda\rho\sin(\omega t-\theta)\}.$$

We obtain the instantaneous forms from the imaginary part of Eq. (15). That is,

$$f_h=\alpha\rho\sin(\omega t-\theta)+\beta\rho\cos(\omega t-\theta),$$

$$\ddot{v}+2\lambda\dot{v}+k^2v=\varepsilon\alpha\rho\sin(\omega t-\theta)+\varepsilon\beta\rho\cos(\omega t-\theta). \tag{16}$$

Furthermor,

$$\dot{v} = \text{imaginary part of } \dot{V} = \text{imaginary part of } \{\dot{\rho} - j(\dot{\theta} - \omega)\rho\}\varepsilon^{j(\omega t-\theta)}$$
$$= \dot{\rho}\sin(\omega t - \theta) - (\dot{\theta} - \omega)\rho\cos(\omega t - \theta)$$
$$= (\dot{v}_m\varepsilon^{-\lambda t} - \lambda v_m\varepsilon^{-\lambda t})\sin(\omega t - \theta) - (\dot{\theta} - \omega)\rho\cos(\omega t - \theta).$$

We assume that the amplitude and phase are slowly varying functions of t, and we neglect the differential terms. Then, we obtain as follows:

$$\dot{v} = -\lambda v_m\varepsilon^{-\lambda t}\sin(\omega t - \theta) + \omega\rho\cos(\omega t - \theta).$$

We combine this equation with Eq. (16) and obtain the following relations:

$$f_h = \alpha v + \frac{\beta}{\omega}(\dot{v} + \lambda v),$$
$$\ddot{v} + (2\lambda - \frac{\varepsilon\beta}{\omega})\dot{v} + (k^2 - \varepsilon\alpha - \frac{\varepsilon\beta\lambda}{\omega})v = 0. \quad (17)$$

Eqs. (15), (17) represent the extended harmonic approximation respectively.

We calculate V, $\dot{V}$, $\ddot{V}$ by the above equation (see paragraph 2) and substitute those into Eq. (15) and equate respectively the resulting real and imaginary part of the equation. Then, we know that trajectory of the extended harmonic approximation in ρ-θ plane is given as follows:

$$\frac{\dot{\rho}}{\rho} = -\lambda + \frac{\varepsilon\beta(\rho)}{2\omega},$$
$$\dot{\theta} = \frac{\varepsilon\alpha(\rho)}{2\omega}. \quad (18)$$

Where the following approximations are assumed:

$$\dot{\rho} = \dot{v}_m\varepsilon^{-\lambda t} - \lambda v_m\varepsilon^{-\lambda t} \fallingdotseq -\lambda\rho,$$
$$\ddot{\rho} = \ddot{v}_m\varepsilon^{-\lambda t} - 2\lambda\dot{v}_m\varepsilon^{-\lambda t} + \lambda^2 v_m\varepsilon^{-\lambda t} \fallingdotseq \lambda^2\rho,$$
$$\rho\ddot{\theta} \fallingdotseq 0, \quad \rho\dot{\theta}^2 \fallingdotseq 0, \quad 2\dot{\rho}\dot{\theta} \ll 2\omega\rho,$$

$\therefore$

$$\begin{cases} V = \rho\varepsilon^{j(\omega t-\theta)}, \\ \dot{V} = (-\lambda\rho + j\omega\rho)\varepsilon^{j(\omega t-\theta)}, \\ \ddot{V} = (\lambda^2\rho + 2\omega\rho\dot{\theta} - \omega^2\rho + 2j\omega\dot{\rho})\varepsilon^{j(\omega t-\theta)}. \end{cases}$$

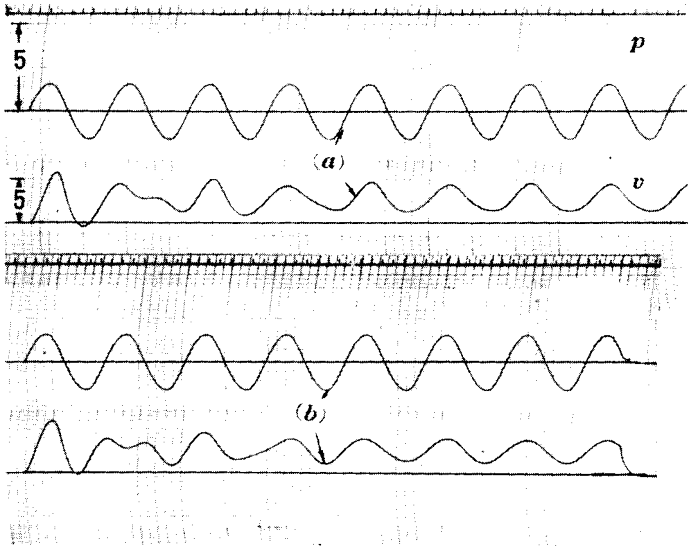

Fig. 16 Comparison between the exact solution (a) and the (0, 1) harmonic approximation (b). (u=0.85)

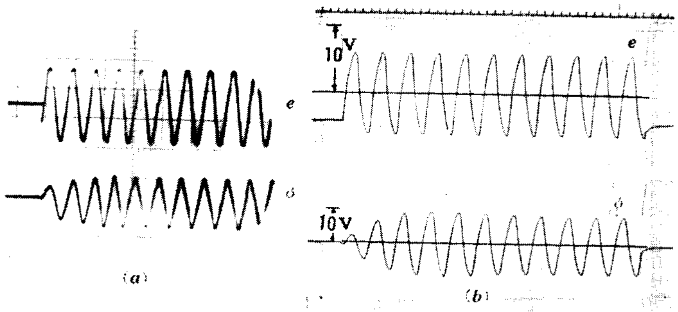

Fig. 17 Comparison between the experimental solution (a) and the harmonic approximation (b). (E=17.3, f=60)

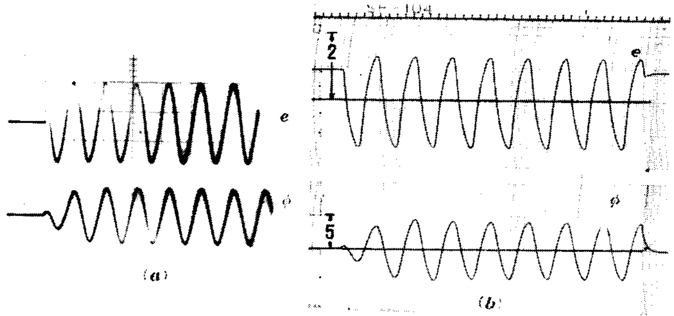

Fig. 18 Comparison between the experimental solution (a) and the harmonic approximation (b). (E=9.37, f=60)

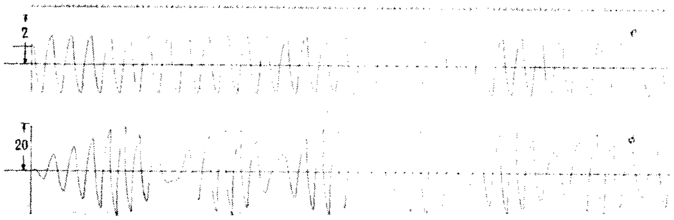

Fig. 19 Solution of the case approximated by the curve of third order term of ϕ and i. (The soltition obtained by a computer with multipliers.)

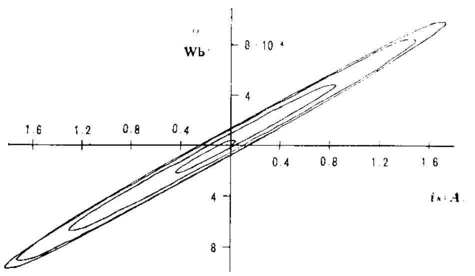

Fig. 20 Hysteresis loop of the harmonic approximation.

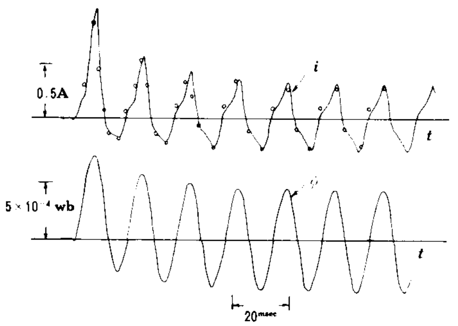

Fig. 21 Comparison between the experimental solution of current and the graphical calculation.

As mentioned above the generating solution is a damped oscillation. Accordingly, even if the coefficient λ of the damping term were considerably large, we could obtain the approximate solution with small error. But, if λ is too large, the phenomenon becomes nonoscillatory. In this case, the present method using the oscillatory generating solution will be inadequate to apply.

§ 7. Conclusions

The essential points of the study may be stated in following items:

(1) The extended harmonic approximation of nonlinear element, and the extended harmonically approximated system of nonlinear system are defined.

(2) When the phenomenon reaches the steady state, it agrees with the harmonic approximation of nonlinear element or the harmonically approximated system of nonlihear system.

(3) Accordingly, at every turn, the present study is the consistent extension of the preceding paper to the transient state. For instance, we can take into consideration not only the fundamental harmonics, but also some other harmonics and a constant term.

(4) The linearization of nonlinear term is first performed. Then, the nonlinear system is transformed into the extended harmonically approximated system, which is expressed by linear differential equations with variable coefficients.

(5) The equations determining the amplitude and phase are obtained from the linear differential equations. The transient solutions of the amplitude and phase are given as solution curves in topological space.

(6) Near the steady solution in phase space the linearization parameters α, β will be sufficient with the first approximation. The system in the first approximation is nothing but the harmonically approximated system.

(7) The transient solution of nonlinear systems is simply obtained with an analogue computer having only linear elements. This solution is not solution curves, but directly gives the oscillation curve.

(8) We do not necessitate special operation for transient solution. We inevitably obtain the approximate transient solution with the steady solution, as the forerunner part of the steady solution by computer operation mentioned in the preceding paper.

(9) The following conditions are required in order that the harmonic approximation in transient state approximates the exact solution.

(a) The nonlinearity is not very great.

(b) The initial value is not situated in too near part of the separatrix.

If those conditions are satisfied, the harmonic approximation can be applied to the so-called pseudo-harmonic or quasi-harmonic oscillation and is very effective.

(10) Even if the relation between the flux and current in transient state are not known previously, the trsansient phenomena of ferroreso-

nant circuit, considering the hysteresis, can be analyzed from the static hysteresis loop.

(11) Making use of the oscillatory generating solution, the present linearization can also be applied to the autonomous system in transient state.

Acknowledgment

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