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# *Asymptotic Theory of Rayleigh Problem in Rarefied Gas*

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Synopsis

The asymptotic theory of Rayleigh shear flow for large values of time is developed on the basis of the linearized Boltzmann-Krook equation. Asymptotic equations for mean velocity outside the Knudsen layer are obtained by employing the Hilbert expansion. Slip boundary conditions are derived from the analysis of the Knudsen layer adjacent to the wall. A solution of the asymptotic equation is obtained under the slip boundary condition and zero initial condition. Discussions are also made of the flow induced by a slowly oscillating flat plate.

## 1. Introduction

The Chapman-Enskog or Hilbert expansion is a basic idea for getting an asymptotic solution of the Boltzmann equation when the characteristic time or length of the flow is much larger than the collision time or mean free path. Asymptotic equations such as Euler, Navier-Stokes and Burnett equations are derived from these expansions. Appropriate initial and boundary values of the mean quantities, i.e., velocity and temperature, must be given for solving the asymptotic equations. The boundary or initial condition is, however, generally specified in terms of the velocity distribution function of the molecules. It is one of the main problems in rarefied gas dynamics to investigate what boundary or initial condition for mean quantities

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should be taken in order to get the correct asymptotic flow subject to the given boundary or initial velocity distribution function. To this end, we must investigate the transition layer, i.e., the Knudsen layer near the boundary or the initial layer, through which the given velocity distribution function relaxes to the near equilibrium distribution function obtained by the Chapman-Enskog or Hilbert expansion. Grad [1] and Sone [2] developed the asymptotic theory of initial value problem without boundaries in the flow. The boundary value problem of steady flow has been studied by Tamada and Yamamoto [3,4], Grad [5], Sone [6,7] and de Wit [8].

Rayleigh shear flow may be a simple example of initial and boundary value problem. There are some studies on the asymptotic behaviour of this flow. Trilling [9] derived the asymptotic equation on the basis of the Boltzmann-Krook equation. However, he neither analyzed the Knudsen layer nor gave a correct boundary condition to the asymptotic equation. Tamada and Sone [10] considered the same problem based on the linearized Boltzmann-Krook equation. They derived an integral equation for the Laplace transform of the flow velocity and obtained a correct asymptotic solution for large times both in and outside the Knudsen layer. Although they give a complete result of the asymptotic behaviour of Rayleigh shear flow, it is not directly concerned with the asymptotic equation for mean velocity and its initial and boundary conditions. In the present paper, we derive the asymptotic equation by applying the Hilbert expansion to the linearized Boltzmann-Krook equation and investigate what boundary and initial conditions should be imposed on the equation to get the correct asymptotic flow for large times.

## 2. Fundamental Equations

We take a semi-infinite expanse of rarefied gas bounded by a plane wall. The gas is at rest with a density $\rho_0$ and a temperature $T_0$. The wall is set in motion with a speed $U(t)(2RT_0)^{1/2}$, where R is the gas constant, in its own plane at $t = 0$. The wall temperature remains unchanged ($T_0$). It is assumed that the gas obeys the Boltzmann-Krook equation [11] and that the gas molecules are reflected diffusely at the wall. We further assume that the wall speed is so small that the equation and the initial and boundary conditions can be linearized. Then, it can be shown that the density and temperature hold their initial states for $t > 0$, while the tangential flow velocity satisfies the following equations [10]:

$$\lambda^{-1/2}\frac{\partial\phi}{\partial t} + v\frac{\partial\phi}{\partial Y} = \lambda^{1/2}(-\phi + 2uq), \tag{1}$$

$$q = \int u\phi E d\underline{v}, \tag{2}$$

$$E = \pi^{-3/2}e^{-\underline{v}^2}, \tag{2a}$$

where $n_o(2RT_o)^{-3/2}E(1+\phi)$ is the velocity distribution function, $n_o$ the number density, $(2RT_o)^{1/2}q$ the flow velocity, $(2RT_o)^{1/2}\underline{v} = (2RT_o)^{1/2}(u, v, w)$ the molecular velocity, $\lambda$ the collision frequency related to the viscosity $\mu$ by $\lambda = (\rho_o RT_o)/\mu$ and $(2RT_o/\lambda)^{1/2}Y$ the normal coordinate. The boundary conditions are given by

$$\phi(v > 0) = 2uU(t) \quad \text{at} \quad Y = 0, \tag{3}$$

$$\phi \to 0 \quad \text{as} \quad Y \to \infty, \tag{4}$$

while the initial condition is taken to be

$$\phi = 0 \quad \text{at} \quad t = 0. \tag{5}$$

The shear stress is expressed by $\phi$ in the form

$$P_{xy} = -2\rho_o RT_o \int uv\phi E d\underline{v}. \tag{6}$$

Under these, we investigate the asymptotic behaviour of the flow for large times after the wall starts to move.

## 3. Asymptotic Field

First, we consider the flow of the asymptotic field where the space and time derivatives are of order one. A solution of Eq.(1) may be obtained by the Hilbert expansion. That is, we expand the quantities in series of $\lambda^{-1/2}$ as follows:

$$\begin{pmatrix}\phi\\ q\end{pmatrix} = \begin{pmatrix}\phi_H\\ q_H\end{pmatrix} = \begin{pmatrix}\phi_H^o\\ q_H^o\end{pmatrix} + \lambda^{-1/2}\begin{pmatrix}\phi_H^1\\ q_H^1\end{pmatrix} + \lambda^{-1}\begin{pmatrix}\phi_H^2\\ q_H^2\end{pmatrix} + \cdots . \tag{7}$$

Putting this into Eq.(1) and equating the same order terms in $\lambda$, we have

$$\left.\begin{aligned}
\phi_H^o &= 2uq_H^o, \\
\phi_H^1 &= 2uq_H^1 - v\frac{\partial\phi_H^o}{\partial Y}, \\
\phi_H^i &= 2uq_H^i - v\frac{\partial\phi_H^{i-1}}{\partial Y} - \frac{\partial\phi_H^{i-2}}{\partial t} \qquad (i = 2,3,\cdots).
\end{aligned}\right\} \tag{8}$$

Substitution of Eq.(8) into Eq.(2) leads to the following asymptotic equations for $q_H^i$:

$$\frac{\partial q_H^i}{\partial t} = \frac{1}{2}\frac{\partial^2 q_H^i}{\partial Y^2} \qquad (i = 0,1), \tag{9}$$

$$\frac{\partial q_H^i}{\partial t} = \frac{1}{2}\frac{\partial^2 q_H^i}{\partial Y^2} + \frac{1}{4}\frac{\partial^4 q_H^{i-2}}{\partial Y^4} \qquad (i = 2,3). \tag{10}$$

$$\cdots\cdots$$

The first two equations are the Stokes equation. Equation (10) includes a term of the fourth order derivative which is to be determined in a previous (lower order) analysis.

The distribution function (7) accompanied by (8) does not generally satisfy the boundary condition (3). There appears a transition layer (Knudsen layer) near the wall whose thickness is of the order of mean free path and in which the distribution function varies rapidly so that it satisfies the boundary condition (3). We next proceed to the analysis of the Knudsen layer.

## 4. Knudsen Layer and Boundary Conditions for $q_H^i$

To analyze the Knudsen layer, we introduce a stretched coordinate $\eta$ which is related to Y by $\eta = \sqrt{\lambda}Y$. Further, we assume the solution of the Knudsen layer in the following forms:

$$\left.\begin{aligned}
\phi &= \phi_H(Y, t) + \phi_K(\eta, t), \\
q &= q_H(Y, t) + \phi_K(\eta, t).
\end{aligned}\right\} \tag{11}$$

The correction terms $\phi_K$ and $q_K$ are also expanded in series of $\lambda^{-1/2}$ as:

$$\begin{pmatrix}\phi_K\\ q_K\end{pmatrix}=\begin{pmatrix}\phi_K^o\\ q_K^o\end{pmatrix}+\lambda^{-1/2}\begin{pmatrix}\phi_K^1\\ q_K^1\end{pmatrix}+\lambda^{-1}\begin{pmatrix}\phi_K^2\\ q_K^2\end{pmatrix}+\cdots \quad . \tag{12}$$

Substituting this into Eqs.(1) and (2), we get the following equations:

$$v\frac{\partial\phi_K^i}{\partial\eta}=-\phi_K^i+2uq_K^i \qquad (i=0,1), \tag{13}$$

$$v\frac{\partial\phi_K^i}{\partial\eta}=-\phi_K^i+2uq_K^i-\frac{\partial\phi_K^{i-2}}{\partial t} \qquad (i=2,3), \tag{14}$$

.....

$$q_K^i=\int u\phi_K^i E d\underline{v} \ . \tag{15}$$

We need only boundary conditions to solve these integro-differential equations (we do not need an initial condition). From Eqs.(3), (11) and (12), the boundary conditions for $\phi_K^i$ at $\eta = 0$ are given by

$$\phi_K^o(v>0)=2uU(t)-\phi_H^o \ , \tag{16}$$

$$\phi_K^i=-\phi_H^i \qquad (i=1,2,\cdots\cdots). \tag{17}$$

As $\eta \to \infty$, the correction terms should vanish

$$\phi_K^i\ ,\ q_K^i \to 0 \ . \tag{18}$$

Solving Eqs.(13) and (14) formally under the conditions (16) $\sim$ (18) and inserting the results into Eq.(15), we obtain the following integral equations for $q_K^i$:

$$L(q_K^o)=\{U(t)-q_H^o(0,\ t)\}J_o(\eta), \tag{19}$$

$$L(q_K^1)=-q_H^1(0,\ t)J_o(\eta)+\left(\frac{\partial q_H^o}{\partial Y}\right)_o J_1(\eta), \tag{20}$$

$$L(q_K^2)=\{-q_H^2(0,\ t)+\frac{1}{2}\left(\frac{\partial^2 q_H^o}{\partial Y^2}\right)_o+\frac{\partial}{\partial t}(U-q_H^o(0,\ t))\}J_o(\eta)$$

$$+\left(\frac{\partial q_H^1}{\partial Y}\right)_o J_1(\eta)-\left(\frac{\partial^2 q_H^o}{\partial Y^2}\right)_o J_2(\eta) \ , \tag{21}$$

$$L(q_K^3) = -q_H^3(0, t)J_o(\eta) + \left(\frac{\partial q_H^2}{\partial Y}\right)_o J_1(\eta) - \left(\frac{\partial^2 q_H^1}{\partial Y^2}\right)_o J_2(\eta) + \left(\frac{\partial^3 q_H^o}{\partial Y^3}\right)_o J_3(\eta) , \tag{22}$$

where

$$L(q_K^i) = \sqrt{\pi} q_K^i - \int_0^\infty q_K^i(\eta_o, t) J_{-1}(|\eta - \eta_o|) d\eta_o , \tag{23}$$

$$J_n(\eta) = \int_0^\infty \xi^n \exp( - \xi^2 - \eta/\xi) d\xi , \tag{24}$$

and the notation $(F)_o$ means the value of F evaluated at Y = 0. In the above equations, we have changed the time derivatives of $q_H^i$ to the space derivatives by use of Eqs.(9) and (10). A standard form of the present integral equations is

$$L(Y_i) = -J_{i+1}(\eta) - \kappa_i J_o(\eta) . \tag{25}$$

Tamada and Sone give approximate numerical values of $Y_i(\eta)$ and $\kappa_i$ [see Ref.10]. By using their results, the solutions of our integral equations are written as

$$q_K^o = 0 , \tag{26}$$

$$q_K^1 = -\left(\frac{\partial q_H^o}{\partial Y}\right)_o Y_o(\eta) , \tag{27}$$

$$q_K^2 = \left(\frac{\partial^2 q_H^o}{\partial Y^2}\right)_o Y_1(\eta) - \left(\frac{\partial q_H^1}{\partial Y}\right)_o Y_o(\eta) , \tag{28}$$

$$q_K^3 = -\left(\frac{\partial^3 q_H^o}{\partial Y^3}\right)_o Y_2(\eta) + \left(\frac{\partial^2 q_H^1}{\partial Y^2}\right)_o Y_1(\eta) - \left(\frac{\partial q_H^2}{\partial Y}\right)_o Y_o(\eta) , \tag{29}$$

together with the following relations between $q_H^i$ and its derivatives at Y = 0:

$$q_H^o(0, t) = U(t) , \tag{30}$$

$$q_H^1(0, t) = -\left(\frac{\partial q_H^o}{\partial Y}\right)_o \kappa_o , \tag{31}$$

$$q_H^2(0,\ t) = \left(\frac{\partial^2 q_H^o}{\partial Y^2}\right)_o (\kappa_1 + \frac{1}{2}) - \left(\frac{\partial q_H^1}{\partial Y}\right)_o \kappa_o \ , \tag{32}$$

$$q_H^3(0,\ t) = -\left(\frac{\partial^3 q_H^o}{\partial Y^3}\right)_o \kappa_2 + \left(\frac{\partial^2 q_H^1}{\partial Y^2}\right)_o \kappa_1 - \left(\frac{\partial q_H^2}{\partial Y}\right)_o \kappa_o \ . \tag{33}$$

These equations (30) $\sim$ (33) constitute the boundary conditions for the asymptotic equations (9) and (10). Equation (30) is the non-slip boundary condition. The first slip condition (31) is the same as in a steady flow. The terms including more than first derivatives appear in the second and third order slip conditions (32) and (33). It will be seen from Eqs.(9) and (10) that these higher order derivatives vanish in a steady flow.

The other condition for $q_H^i$ is given at infinity. From Eq.(4), it becomes

$$q_H^i(Y,\ t) \to 0 \qquad (Y \to \infty). \tag{34}$$

We can calculate the flow inside the Knudsen layer from Eqs.(27) $\sim$ (29) provided that the asymptotic velocity $q_H^i$ is obtained.

## 5. Asymptotic Solution and Shear Stress

We now proceed to solve the asymptotic equations (9) and (10) under the boundary conditions (30) $\sim$ (34). We need one more condition, namely, an initial condition to complete the system of the equations. Considering that the distribution function is absolutely Maxwellian with zero velocity at the initial stage of the flow, we may take

$$q_H^i(Y,\ 0) = 0 \ . \tag{35}$$

It is easy to solve Eqs.(9) and (10) together with (30) $\sim$ (35) by applying the Laplace transformation when U(t) is taken to be a constant $U_o$. Only the result is given here for shortness:

$$q_H^o/U_o = \mathrm{Erfc}(\zeta)\ , \tag{36}$$

$$q_H^1/U_o = \frac{\sqrt{2}\kappa_o}{\sqrt{\pi t}} \exp(-\ \zeta^2)\ , \tag{37}$$

$$q_H^2/U_o = \frac{\zeta}{\sqrt{\pi t}} (\zeta^2 + 2\kappa_o^2 + 2\kappa_1 + \frac{1}{2}) \exp(-\ \zeta^2)\ , \tag{38}$$

$$q_H^3/U_o = \sqrt{\frac{2}{\pi}}\, t^{-3/2} \{-\kappa_o(\frac{3}{2} - \zeta^2)\zeta^2 + (\kappa_o^3 + 2\kappa_o\kappa_1 + \kappa_2 + \frac{\kappa_o}{4})(2\zeta^2 - 1)\}\exp(-\zeta^2), \tag{39}$$

where $\zeta = Y/\sqrt{2t}$ and

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\eta^2} d\eta .$$

Tamada and Sone [10] treat the same flow by a different way of analysis. They solved the linearized Boltzmann-Krook equation directly to get the distribution function by applying the initial and boundary conditions to it. Then, they derived an integral equation for the flow velocity, on which they discussed the asymptotic behaviour of the flow for large values of time. Comparing our solution (36) ∿ (39) with theirs carefully, we will see that both results are the same. This seems to indicate that our method of analysis, especially, the boundary and initial conditions result a correct solution.

It is easy to calculate the correction of the Knudsen layer by use of the solution obtained above and equations (27) ∿ (29). We here add the formula of the shear stress. Putting $\phi_H$ and $\phi_K$ obtained in the preceeding sections into Eq.(6), we can calculate the shear stress. For instance, at the wall (Y = 0) it is given by

$$\frac{P_{xy}\sqrt{\lambda}}{\rho_o RT_o} = \left(\frac{\partial q_H^o}{\partial Y}\right)_o + \lambda^{-1/2}\left(\frac{\partial q_H^1}{\partial Y}\right)_o + \lambda^{-1}\left\{\left(\frac{\partial q_H^2}{\partial Y}\right)_o + \left(\frac{\partial^3 q_H^o}{\partial Y^3}\right)_o \left(\frac{1}{2} + \int_0^\infty Y_o(\eta)d\eta\right)\right\} + \cdots . \tag{40}$$

On using Eqs.(36) ∿ (38) and the numerical value $\int_0^\infty Y_o(\eta)d\eta = 0.2333$, we have

$$\frac{P_{xy}\sqrt{\lambda}}{\rho_o RT_o} = \left\{- 1 + \frac{0.750}{\lambda t}\right\}\sqrt{\frac{2}{\pi t}}\, U \tag{41}$$

This is also the same formula as is given by Tamada and Sone.

## 6. Oscillating Plate

From the preceeding analysis, it will be seen that the boundary conditions (30) ∿ (33) are valid not only when the wall velocity is constant, but also when it varies in time. As a simple example of the latter case, we consider the flow induced by an oscillating infinite plate in its own plane. We take the wall speed to be

$$U(t) = U_o \cos \omega t \ , \tag{42}$$

and analyze the case when the frequency of oscillation is much smaller than the collision frequency ($\omega \ll \lambda$) by applying the present asymptotic theory. We can easily obtain a solution and give here only the result. The asymptotic velocity is given by

$$q_H^o = U_o e^{-\sqrt{\omega}Y} \cos (\omega t - \sqrt{\omega}Y) \ , \tag{43}$$

$$q_H^1 = U_o \kappa_o \sqrt{2\omega}\ e^{-\sqrt{\omega}Y} \cos (\omega t - \sqrt{\omega}Y + \frac{\pi}{4}) \ , \tag{44}$$

$$q_H^2 = U_o (2\omega)\ e^{-\sqrt{\omega}Y} \{(\kappa_o^2 + \kappa_1 + \frac{1}{2}) \cos (\omega t - \sqrt{\omega}Y + \frac{\pi}{2}) + \frac{\sqrt{2\omega}}{4} Y \cos (\omega t - \sqrt{\omega}Y + \frac{3}{4}\pi)\} \ , \tag{45}$$

$$q_H^3 = U_o (2\omega)^{3/2}\ e^{-\sqrt{\omega}Y} \{(\kappa_o^3 + 2\kappa_o\kappa_1 + \kappa_2 + \frac{\kappa_o}{4}) \cos (\omega t - \sqrt{\omega}Y + \frac{3}{4}\pi) + \kappa_o \frac{\sqrt{2\omega}}{4} Y \cos (\omega t - \sqrt{\omega}Y + \pi)\} \ . \tag{46}$$

Using these results and the formula (40), we can calculate the shear stress and obtain as

$$-\sqrt{\frac{\lambda}{2\omega}} \frac{P_{xy}}{\rho_o R T_o U_o} = \cos (\omega t + \frac{\pi}{4}) + \sqrt{\frac{2\omega}{\lambda}} \kappa_o \cos (\omega t + \frac{\pi}{2}) + \frac{2\omega}{\lambda} \left[\kappa_o^2 + \kappa_1 + \frac{3}{4} + \int_0^\infty Y_o(\eta) d\eta\right] \cos (\omega t + \frac{3}{4}\pi) \ . \tag{47}$$

According to Ref.10, the numerical values of $\kappa_i$ are given by

$$\kappa_o = -1.016 \ , \quad \kappa_1 = -1.266 \ , \quad \kappa_2 = -1.820$$

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