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Estimation of the Number of Iterations for Detinite Convergence Condition by Use of the Gauss-Seidel Method

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SYNOPSIS

In this investigation the estimation method of the number of iterations for definite convergence condition by use of the Gauss-Seidel method applied for a set of linear equations which is obtained from the finite element analysis (or the finite difference analysis) of any rectangular area subdivided into N*M is proposed. Though the number of iterations can be obtained by using the eigenvalue of the governing equations, the proposed method does not require the eigenvalue but only the values of N and M. Numerical experiments on this estimation method clarify that the estimated values are within the error bound of 10%.

1. INTRODUCTION

The Gauss-Seidel Method which is one of iterative methods for linear equations is used in wide engineering fields because of its steady convergence characteristic. On the contrary, by the reason of its slow convergence its user oftenly fails to obtain enoughly convergent solution and the user can obtain too poor approximate solution. This is mainly caused by the reason that the number of iterations of the Gauss-Seidel method for the present problem cann't be estimated beforehand.

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the iterative solvers are sufficiently done, and it is already clarified that the convergence ratio is measured by use of the matrix norm of the coefficient matrix of the governing equations[1,2]. By use of the relation between the matrix norm and the maximum eigenvalue we can obtain the lower bound of the neccessary number of iterations. But, for this estamation we have to obtain, of course, the maximum eigenvalue of the governing equation.

The main purpose of this investigation is to propose an estimation method of the number of iterations of the Gauss-Seidel method which does not require the computation of the eigenvalue, and the method is obtained from the results of numerical experiments.

2. CONVERGENCE CHARACTERISTICS OF THE GAUSS-SEIDEL METHOD

Let

$$Ax = b \tag{1}$$

be a set of linear equations, where A is a (L*L) coefficient matrix, and x and b are unknown and known vectors, respectively.

From the i-th equation of eq.1 we obtain

$$x_i = (b_i - \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{L} a_{ij}x_j)/a_{ii} \tag{2}$$

By giving an appropriately selected initial value for x, say $x^{(0)}$, eq.2 gives new solution vector $x^{(1)}$. By repeating this procedure $x^{(k+1)}$ is expressed as following;

$$x_i^{(k+1)} = (b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{L} a_{ij}x_j^{(k)})/a_{ii} \tag{3}$$

In eq.3 $x_j^{(k)}$ for $j<i$ may be replaced by $x_j^{(k+1)}$, because $x_j^{(k+1)}$ are already obtained at the stage. Then,

$$x_i^{(k+1)} = (b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{L} a_{ij}x_j^{(k)})/a_{ii} \tag{4}$$

Eq.4 is a general expression of the Gauss-Seidel method.

Let D, E and F be the main diagonal, lower triangular and upper triangular matrix of A, respectively. That is,

$$A = D - E - F \tag{5}$$

The matrix representation of eq.4 is

$$x^{(k+1)} = D^{-1}(b + Ex^{(k+1)} + Fx^{(k)})$$
$$Dx^{(k+1)} = b + Ex^{(k+1)} + Fx^{(k)}$$

∴

$$x^{(k+1)} = (D - E)^{-1} F x^{(k)} + (D - E)^{-1} b \quad (6)$$

If $(D - E)^{-1}F$ and $(D - E)^{-1}b$ are expressed by C and g, respectively, then eq.6 is replaced by eq.7.

$$\begin{aligned} x^{(k+1)} &= Cx^{(k)} + g \\ &= C(Cx^{(k-1)} + g) + g \\ &\cdots\cdots \\ &= C^k x^{(1)} + (C^{k-1} + C^{k-2} + \ldots + C^2 + C + I)g \end{aligned} \quad (7)$$

, where I is a unit matrix, and C is called as the Point Gauss-Seidel Matrix[1].

Here, we introduce the definition of a matrix norm expressed as following;

$$\| B \| = \sup_{x \neq 0} \frac{\| Bx \|}{\| x \|} \quad (8)$$

$\| B \|$ is the spectral norm of the matrix B [1]. By introducing eq.8 into eq.7 we obtain that if $\| C \| < 1$ for $k \to \infty$, then the solution vector tends to a convergent vector expressed by

$$x = (I - C)^{-1} g \quad (9)$$

That is, if $\| C \| = \| (D - E)^{-1} F \|$ is less than 1, the Gauss-Seidel method converges to a vector X.

Let $\varepsilon^{(n)}$ and $\varepsilon^{(m)}$ be error vectors after n and m iterations of the Gauss-Seidel method applied to eq.1.

$$\begin{aligned} \varepsilon^{(n)} &= x^{(n)} - x \\ \varepsilon^{(m)} &= x^{(m)} - x \end{aligned} \quad (10)$$

, where x is a strict solution vector. For $n > m$, we assume following relation for these two error vectors;

$$\frac{\| \varepsilon^{(n)} \|}{\| \varepsilon^{(m)} \|} = \frac{1}{10} \quad (11)$$

From eq.7,

$$\begin{aligned} \varepsilon^{(n)} &= \| C\varepsilon^{(n-1)} \| \leqslant \| C \| \; \| \varepsilon^{(n-1)} \| \\ &\leqslant \| C \|^2 \; \| \varepsilon^{(n-2)} \| \\ &\quad \vdots \\ &\leqslant \| C \|^{n-m} \; \| \varepsilon^{(m)} \| \end{aligned} \quad (12)$$

Substitution of eq.12 into eq.11 leads to following expression.

$$\| C \|^{n-m} \geqslant 1/10 \quad (13)$$

On the other hand, the spectral norm of C is related to the spectral radius $\rho(C)$ as following;

$$\| C \| \geqslant \rho(C) \tag{14}$$

Since the spectral norm is equal to the maximum eigenvalue of C, that is,

$$\rho(C) = \max_{1 \leqslant i \leqslant L} | \lambda_i | = \lambda_{max} \tag{15}$$

, then

$$(\lambda_{max})^{\delta IT} \geqslant 1/10 \tag{16}$$

, where IT = n - m.
From eq.16 we obtain

$$\delta IT > - \frac{1}{\log \lambda_{max}} \tag{17}$$

Eq.17 indicates that the number of iterations in order to obtain a better approximate solution vector by one figure is decided by the maximum eigenvalue of the Point Gauss-Seidel matrix C.

Let m, m' and m'' be the figures of initial value, strict solution and the final allowable error of the Gauss-Seidel method, respectively. From the linear convergence characteristic of the Gauss-Seidel method it is obvious that the upper bound of the neccessary number of iterations to adjust the approximate solution to the first figure of the strict solution and to adjust the solution vector within the allowable error bound are estimated as following, respectively;

$$IT(1) \leqslant | m - m' | \delta IT$$

$$IT(2) \leqslant | m' - m'' | \delta IT$$

Then, the total number of iterations is estimated as the sum of them.

$$IT = IT(1) + IT(2)$$

$$\leqslant (| m - m' | + | m' - m'' |) \delta IT \tag{18}$$

Eq.18 indicates that the neccessary number of iterations is governed by the number of iterations for adjusting one figure, i.e. δIT, the accuracy of the initial value and the allowable error bound. Among these three factors the latter two are rather easily estimated from the engineering point of view, but the first factor is difficult to decide, because it is governed by the maximum eigenvalue of the matrix as shown in eq.17.

Our aim of this investigation is the estimation of δIT without the computation of the maximum eigenvalue of the point Gauss-Seidel matrix C.

3. NUMERICAL EXPERIMENTS

3.1 MAIN FACTORS DECIDING THE NUMBER OF ITERATIONS

From the theoretical investigations on the number of iterations of the Gauss-Seidel method required for the better approximate solution by one figure it is clarified that the value δIT is governed by the maximum eigenvalue of the point Gauss-Seidel matrix C;

$$C = (D - E)^{-1}F$$

If we may allow to compute the eigenvalue of C matrix, then δIT is obtained by using eq.17. But, generally speaking, the computation of the eigenvalue requires rather long execution-time. Therefore, we aim to estimate δIT directly from the results of a number of numerical experiments.

At the beginning we assume that the objective linear equations treated here are obtained from the finite element method applied to a rectangular area subdivided into N*M.

The coefficient matrix A of the linear equations is generally sparse, and A is determined by following factors:

1. Physical properties
2. Topological properties
3. Boundary conditions

Factor 1 includes the geometry, load intensity, stiffness of element, the position of loads, and so on, Factor 2 includes the number of nodes and elements, the differences of mesh patterns, and so on, and Factor 3 indicates the differences of the restriction of nodes.

In order to clarify which factor among them governs δIT a number of numerical experiments are done by changing the combination of these factors. The procedure is as following;

Step 1. Give the initial value x=0 and the final error bound E_f. Set K=1.

Step 2. Calculate the error bound, $E = 0.1^K$.

Step 3. Set d = 1.

Step 4. Calculate the solution vector $x^{(d)}$ by use of the Gauss-Seidel method.

Step 5. If $| x^{(d)} - x^{(d-1)} | > E$ for k = 1,2, ... , L, then go to Step 7. Otherwise, go to Step 6.

Step 6. Set d = d + 1, and go to Step 4.

Step 7. Print out "d" and "E". If $E \neq E_f$, set K = K + 1 and go to Step 2.

In above procedure "d" indacates the value of δIT. The propriety of the judgement method of the convergence (Step 5) is confirmed by the results of numerical experiments shown in Table 1 in Appendix, in which three methods including above judgement are compared and it clarifies that there are no significant difference between them.

(1) Factor 1

The results of the numerical experiments for Factor 1 are summarized in Table 2,3 and 4 in Appendix. From these tables we may conclude that Factor 1 has no influence to the value of δIT.

(2) Factor 2

As the factors included in this category there are i) the number of nodes, ii) connectivity between nodes (or mesh pattern) and iii) the combination of N and M for the same number of nodes L (or the shape of the rectangular area). The results are illustrated in Table 5,6 and 7, respectively. Note that the mesh pattern considered in this experiments are selected from the meshes commonly used in the finite element analysis. These experiments clarify that the number of nodes and the shape of the rectangular area give influence to the value of δIT but the difference of the mesh patterns is less important for the value.

(3) Factor 3

Since any rectangular area is surrounded by four edges, following five types of boundary conditions must be considered:

Type 1; Four edges are fixed.

Type 2; Three edges are fixed.

Type 3; Adjacent two edges are fixed.

Type 4; Opposite two sides are fixed.

Type 5; Only one edge is fixed.

The result given in Table 8 in Appendix indicates that the difference of the boundary condition is one of the most important factors to decide δIT.

From above experiments we may conclude that following three factors give influence to the value of δIT:

1. The number of nodes, L.
2. The shape of the rectangular area, N and M.
3. The boundary condition.

Therefore, we may investigate the influence of these factors to the value of δIT.

3.2 NUMERICAL EXPERIMENTS FOR ONE-DIMENSIONAL SYSTEM

One-dimensional system treated in this section is a system that all nodes are connected in a series. Since the physical values in the system has less significant for the value of δIT, the connectivity between every neighbouring nodes is estimated as "1", that is $a_{ij}=-1$ for $i \neq j$, and $a_{ii}= \Sigma|a_{ij}|$ for j=1 to L. For the node fixed to the boundary "1" is added to a_{ii}. As the load vector, b, unit load vector is applied, that is $b_i=1$ for all "i".

As the test example a simple path with 50 nodes is treated as shown in Table 9 in Appendix, and for 18 different boundary conditions the numerical experiments are done. From the table it becomes obvious that δIT neccessary for raising the accuracy by one figure is stable for any case, and furthermore, the value is related to the maximum distance, d_{max}, from any node to the boundary as following;

$$\delta IT \propto d_{max}{}^2 \qquad (19)$$

3.3 NUMERICAL EXPERIMENTS FOR RECTANGULAR AREA SUBDIVIDED INTO N*M

Examples treated here are rectangular area subdivided into N by M, and the order of A matrix is L (= N*M). Physical values are just same as described in Section 3.2. For five types of the boundary conditions mentioned already the numerical experiments are done, and the results are summarized in Tables 10 to 14, respectively. In the tables L, H, B and k indicate the number of nodes, the length of the vertical and lateral edges, and the value of H/B, respectively. The other columns in tables are the results of the estimation method of δIT which is explained in Section 4.

From these results following items are easily found out:

(1). The maximum distance to the boundary seems to decide δIT.

(2). For systems with same number of nodes and the same boundary condition k (=H/B) decides the value.

(3). For the systems with the same number of nodes and same k the boundary condition decides the value.

Another important fact is recognized from the numerical experiment for (N*N) systems with Type 1 boundary condition. That is, for any system with above conditions we can estimate δIT by following simple equation;

$$\delta IT \doteq L^2/4 \qquad (20)$$

4. ESTIMATION METHOD OF δIT

The purpose of this section is the proposal of the estimation method of δIT which is applicable for any rectangular area subdivided into N*M. The method is led from the results of the numerical experiments for the area whose four edges are fixed to the boundary.

Let's assume that the maximum distance from the boundaries decides the value of δIT, since this fact is recognized from the numerical experiments for the one-dimensional system. Then,

$$\delta IT \propto d_{max}^2 \qquad (21)$$

Fig.1 shows a typical rectangular area subdivided into N*M (= (B -1)*(H-1)). We assume B < H. Then, eq.21 becomes as following;

$$\delta IT \propto \left(\frac{B}{2}\right)^2 \qquad (22)$$

If B = H, then δIT estimated by eq.22 must coincide with eq.20. For this purpose, we introduce a parameter γ into eq.22, and eq.22 is replaced by following expression;

$$\delta IT \propto \gamma B * B/2 \qquad (23)$$

That is, δIT is estimated by the number of nodes included in the triangular area in Fig.1.

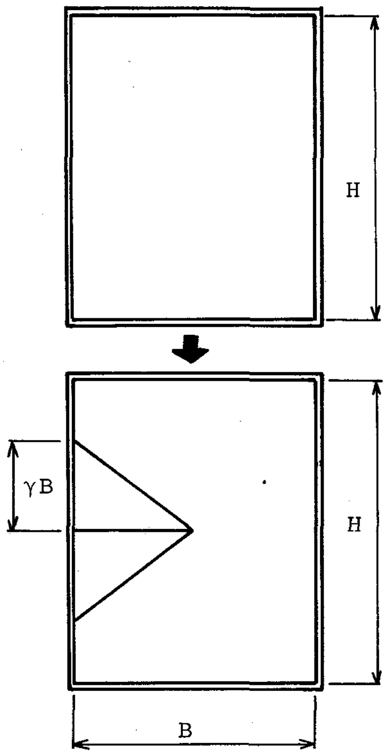

Fig.1

For the case of B = H, γ must be equal to 1/2 from eq.20. And, it is obvious that the value of γ is influenced by the value of k (=H/B) from the numerical experiments.

Then, we introduce following two estimation equations for γ.

$$\gamma = \frac{k}{k+1} \quad (=\gamma_1)$$
$$\gamma = \frac{k^2}{k^2+1} \quad (=\gamma_2) \qquad (24)$$

The results of the δIT-estimations by using above two equations are presented in both columns (δIT_1 and δIT_2) in Table 10. From the table we can recognize that the relation between δIT_1, δIT_2 and actual δIT is expressed by

$$\delta IT_1 < \delta IT < \delta IT_2 \qquad (25)$$

From this result we propose to estimate γ by following equation;

$$\gamma = \frac{\gamma_1 + \gamma_2}{2}$$

$$= \frac{k(k^2 + k + 1)}{2(k + 1)(k^2 + 1)} \qquad (26)$$

Then, the estimation of δIT is obtained by substituting eq.26 into eq.23.

$$\delta IT = \gamma B \frac{B}{2}$$

$$= \frac{k(2k^2 + k + 1)}{4(k + 1)(k^2 + 1)} B^2 \qquad (27)$$

The results of the application of eq.27 are presented in δIT-column in Table 10 in Appendix.

Eq.27 is applicable for systems with other boundary conditions (i.e. Type 2 to 5), if it is replaced to equivalent system with Type 1 bundary condition. The term "equivalent" means that the both systems have the same value of δIT.

How to replace original system to an equivalent one with Type 1 boundary condition is illustrated in Fig's 2 to 5.

<u>1). Type 2 Boundary Condition</u>

If H'<B', then set H=2*H' and B=B', and apply eq.27.(Fig.2a)
If H'>B', then set H=H'+B' and B=B', and apply eq.27.(Fig.2b)

<u>2). Type 3 Boundary Condition</u>

Set H=2*H' and B=2*B', and apply eq.27.(Fig.3)

<u>3). Type 4 Boundary Condition</u>

If H'>4*B', then set H=H' and B=B', and apply eq.27.(Fig.4a)
If H'<4*B', then set H=4*B' and B=B', and apply eq.27. (Fig.4b)

<u>4). Type 5 Boundary Condition</u>

Set B=2*B'. If H'>4*B, then set H=H' and B=2*B', and apply eq.27.(Fig.5a)

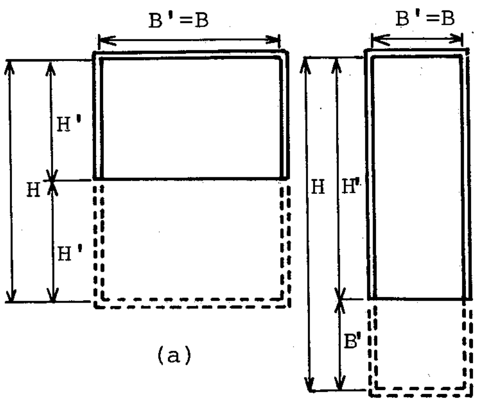

Fig.2

If H'<4*B, then set H=4*B and B=2*B', and apply eq.27. (See Fig.5b)

The estimated values of δIT by above procedure are given in the column indicated by δIT in Tables 11 to 14, respectively.

The comparison of these values with the ones obtained from the numerical experiments clarifies that the difference between them is within 10%. That is, the estimation by use of eq.27 is well agreement with the experiments, and therefore, we may conclude that eq.27 is valid for the evaluation of δIT.

5. CONCLUDING REMARKS

Through these investigations it becomes possible to estimate the number of iterations of the Gauss-Seidel Method applied to any finite element mesh of a rectangular area which is regularily subdivided.

Though the idea of this estimation method comes from the results of the numerical experiments for the one-dimmensional system, that is, δIT depends on the square of the maximum distance from the boundary, eq.27 finally obtained includes the contradiction that it can be applied for two-dimensional systems but not for the one-dimensional one. This indicates that the estimation method of δIt proposed in this

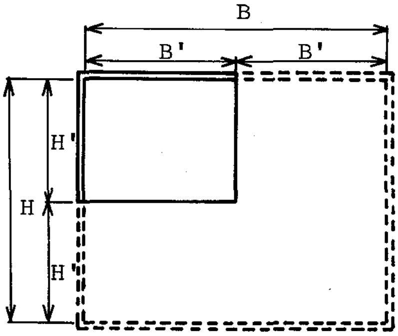

Fig.3

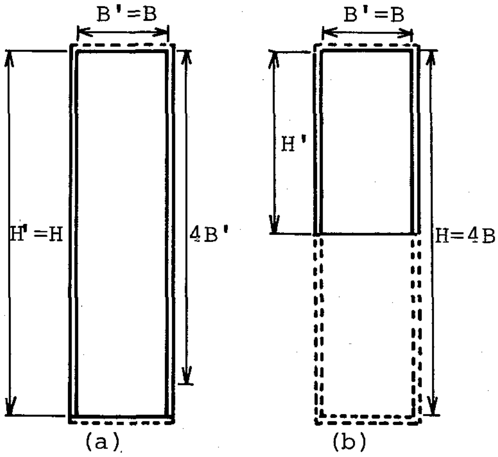

Fig.4

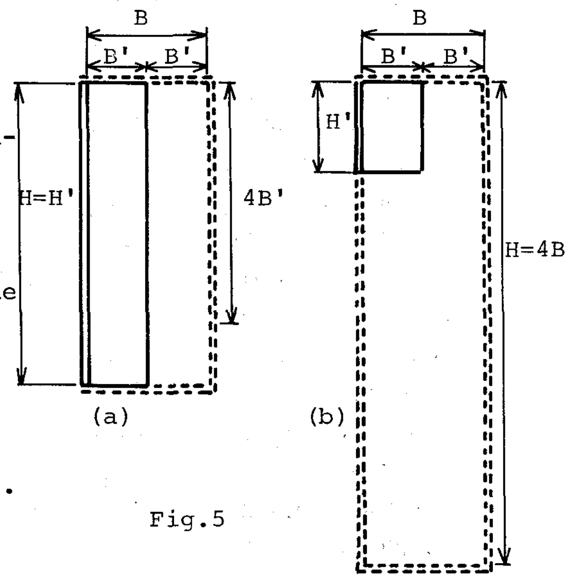

Fig.5

paper is not the final one but only a tentative method, and the final one must accept any kind of linear equations obtained from the application of the finite element method and also the finite diference method.

The estimation of δIT is important for other kinds of engineering problems. As described in Section 2, since δIT is decided by the maximum eigenvalue, the eigenvalue is conversely estimated from δIT. That is, this study leads to the problem of searching the maximum eigenvalue.

REFERENCES

[1] R.S. Varga,'Matrix Iterative Analysis', Prentice-Hall (1962), pp.1-96

[2] D.M. Young,'Iterative Solution of Large Linear Systems', Academic Press (1971), pp.106-139

APPENDIX

In tables following notations are used:

B.C. ; Boundary condition
N ; Number of nodes on the vertical edge
M ; Number of nodes on the lateral edge
L ; Total number of nodes (=N*M)
Judge ; Judgement method of the convergence
δIT ; Number of iterations for getting better result by one figure
E ; Allowable error bound
Mesh ; Mesh pattern
Const. ; Node number connected to the boundary
d_{max} ; Maximum distance to the boundary
H ; Length of the vertical edge
H' ; Original length of the vertical edge
B ; Length of the lateral edge
B' ; Original length of the lateral edge
k ; H/B
k' ; H'/B'
δIT_1 ; Estimated δIT using γ_1
δIT_2 ; Estimated δIT using γ_2

Table 1

Relation between Judgement Method and Number of Iterations

B.C.	N*M	L	Judge	E					
				10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
Type 1	16*16	256	I	63	67	67	67	67	67
			II	10	38	61	67	67	67
			III	82	0	54	66	67	67
Type 4	10*14	140	I	85	100	99	100	100	99
			II	11	45	86	98	100	99
			III	115	0	70	97	99	99
Type 5	8*12	96	I	222	262	262	261	262	262
			II	11	64	188	251	261	261
			III	95	0	65	260	260	260

Table 2

Relation between Stiffness and Number of Iterations

Example	E	
	10^{-5}	10^{-6}
Normal	75	14
(1)*10	74	15
(1)*100	74	16
(2)*10	74	15
(2)*100	74	15
(3)*10	74	14
(3)*1000	74	14

Note : Test example contains 43 nodes. "Normal" indicates all stiffness equal to 1. (1), (2) and (3) indicate that 1, 3 and 7 nodes among 43 are stiffened as shown in the table.

Table 3

Relation between Loading Position and Number of Iterations

Case	E						
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
1	4	51	90	91	90	91	90
2	6	73	91	90	91	90	91
3	7	86	91	90	91	90	91
4	9	93	91	90	91	90	91
5	18	90	91	90	91	90	91
6	22	90	90	91	90	91	90
7	24	90	90	91	90	90	91
8	24	90	90	91	90	90	91

Table 4

Relation between Load Intensity and Number of Iterations

	E						
Case	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
1	79	91	90	91	90	91	90
2	122	91	90	91	90	91	90
3	142	91	90	91	90	91	90
4	156	90	91	90	91	90	91
5	166	90	90	91	90	91	90

Table 5

Relation between Number of Nodes and Number of Iterations

		E				
N*M	L	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
7*7	49	15	15	15	15	15
12*12	144	37	40	39	39	39
16*16	256	63	67	67	67	67
20*20	400	95	102	103	103	102

Table 6

Relation between Mesh Pattern and Number of Iterations

B.C.	N*M	L	Mesh	E					
				10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
Type 1	14*14	196	I	10	32	49	52	52	52
			II	10	33	48	53	52	53
			III	10	33	50	53	54	54
Type 4	16*16	256	I	11	51	110	128	130	130
			II	11	51	109	127	130	130
			III	11	51	109	127	129	130

Table 7

Relation between Configuration(k) and Number of Iterations

B.C.	L	N*M	E				
			10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Type 1	144	12*12	50	5	35	40	40
		18*8	38	8	30	30	30

Table 8

Relation between B.C. and Number of Iterations

B.C.	E				
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Type 1	65	0	50	50	51
Type 2	96	0	60	78	80
Type 3	216	0	56	180	180
Type 4	105	0	72	99	102
Type 5	390	0	40	465	355

Table 9

Relation between Max. Distance and Number of Iterations

Case	Const.	d_{max}	d_{max}^2	E					
				10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
1	25	26	676	750	0	0	375	645	660
2	17,34	17	289	195	0	0	215	275	270
3	1,50	25	625	675	0	0	450	590	605
4	13,25,37	14	196	130	0	20	185	175	185
5	1,17,34,50	9	81	85	0	35	85	90	90

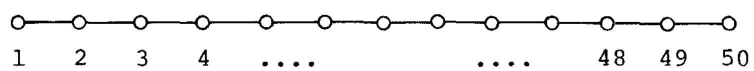

Table 10

Estimation of Number of Iterations for Type 1-B.C.

Case	L	H	B	k	δIT_1	δIT_2	δIT	E				
								10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1	196	15	15	1.00	56	56	56	65	0	54	54	55
2	400	21	21	1.00	110	110	110	125	0	65	100	100
3	144	19	9	2.11	28	33	30	38	8	30	30	30
4	451	42	12	3.50	56	67	61	74	0	58	61	62
5	553	80	8	10.0	29	32	30	34	8	30	29	30
6	600	121	6	20.2	17	18	18	19	9	17	17	17

Table 11

Estimation of Number of Iterations for Type 2-B.C.

Case	L	H'	B'	k'	H	B	k	δIT	E				
									10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1	64	4	17	0.24	8	17	2.13	24	23	6	25	27	26
2	144	8	19	0.42	16	19	1.19	72	86	0	56	68	70
3	196	14	15	0.93	28	15	1.87	80	96	0	60	78	80
4	144	18	9	2.00	27	9	3.00	33	42	6	34	34	34
5	48	12	5	2.44	17	5	3.40	11	14	7	11	10	11
6	64	16	5	3.20	21	5	4.20	11	12	7	10	11	11

Table 12

Estimation of number of Iterations for Type 3-B.C.

Case	L	H'	B'	k'	H	B	k	δIT	E				
									10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1	100	10	10	1.00	20	20	1.00	100	116	0	60	88	96
2	144	12	12	1.00	24	24	1.00	144	170	0	60	130	135
3	196	14	14	1.00	28	28	1.00	196	216	0	56	180	180
4	144	18	8	2.25	36	16	2.25	98	123	0	66	99	102
5	48	12	4	3.00	24	8	3.00	26	35	6	28	28	29
6	64	16	4	4.00	32	8	4.00	28	29	6	27	30	30

Table 13

Estimation of Number of Iterations for Type 4-B.C.

Case	L	H'	B'	k'	H	B	k	δIT	E 10^{-1}	E 10^{-2}	E 10^{-3}	E 10^{-4}	E 10^{-5}
1	116	3	30	0.10	120	30	4.00	392	400	0	59	364	360
2	104	7	14	0.50	56	14	4.00	85	99	0	66	83	86
3	100	9	11	0.82	44	11	4.00	53	63	0	54	51	54
4	99	10	10	1.00	40	10	4.00	44	52	4	44	44	44
5	144	15	10	1.50	40	10	4.00	44	53	3	46	44	45
6	100	24	5	4.80	24	5	4.80	11	14	8	11	12	12
7	123	40	4	10.00	40	4	10.00	8	9	7	7	8	7
8	183	60	4	15.00	60	4	15.00	8	9	7	7	8	7
9	258	85	4	21.25	85	4	21.25	9	9	7	7	8	7

Table 14

Estimation of Number of Iterations for Type 5-B.C.

Case	L	H'	B'	k'	H	B	k	δIT	E 10^{-1}	E 10^{-2}	E 10^{-3}	E 10^{-4}	E 10^{-5}
1	90	2	30	0.07	240	60	4.00	1566	1690	0	0	640	1410
2	80	3	20	0.15	160	40	4.00	696	780	0	0	540	670
3	102	5	17	0.29	136	34	4.00	503	585	0	0	480	495
4	96	7	12	0.58	96	24	4.00	251	95	0	65	260	260
5	196	13	14	0.93	112	28	4.00	341	390	0	40	365	355
6	144	17	8	2.13	64	16	4.00	111	144	0	75	120	-
7	392	48	8	6.00	64	16	4.00	111	142	0	77	124	126
8	129	42	3	14.00	42	6	7.00	17	23	8	19	19	19
9	183	60	3	20.00	60	6	10.00	17	22	7	18	18	19
10	225	74	3	24.67	74	6	12.33	17	57	7	20	20	19