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ON HYPERBOLIC AREA OF THE MODULI OF θ-ACUTE TRIANGLES

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Abstract. A θ-acute triangle is a Euclidean triangle on the plane whose three angles are less than a given constant θ. In this note, we shall give an explicit formula computing the hyperbolic area $A(\theta)$ of the moduli region of θ-acute triangles on the Poincaré disk. It turns out that $A(\theta)$ is a period in the sense of Kontsevich-Zagier if $\cot\theta$ is a nonnegative algebraic number.

1. Introduction

In [4], to each similarity class Δ of triangles on the complex plane $\mathbb{C}$, associated is an invariant $\phi(\Delta)$ valued in the unit disk $\mathcal{D} := \{\zeta \in \mathbb{C}; |\zeta| < 1\}$: If Δ is represented by a triangle $\{a, b, c\} \subset \mathbb{C}$ with $z := \frac{a-b}{c-b}$ ($\mathrm{Im}(z) > 0$), then $\phi(\Delta)$ is defined by

$$\phi(\Delta) := \left(\frac{\rho^2 - \rho z}{z + \rho^2}\right)^3 \qquad (\rho = e^{\frac{2\pi i}{6}}). \tag{1.1}$$

It turns out that the similarity classes of triangles are in one-to-one correspondence with the set of points of $\mathcal{D}$. (See [4] for details and some applications to elementary geometry.)

The purpose of this note is to compute the area $A(\theta)$ of the moduli region of θ-acute triangles

$$M(\theta) := \{\phi(\Delta) \in \mathbb{C} \mid \text{all three angles of } \Delta < \theta\} \tag{1.2}$$

for $\pi/3 < \theta \leq \pi$ measured with the standard hyperbolic (Poincaré) metric of the unit disk $\mathcal{D}$.

We prove the following

Theorem A. *(i) For $\theta > \pi/2$, we have $A(\theta) = \infty$.*

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(ii) For $\pi/3 < \theta \leq \pi/2$, we have

$$\begin{aligned} A(\theta) =&(6\theta - 2\pi) + \frac{k}{\sqrt{3}} \log \frac{4k^2}{k^2+3} \\ &+ \frac{p}{2\beta} \log \left(\frac{1+\sqrt{3}pk+\beta k^2}{1-\sqrt{3}pk+\beta k^2} \cdot \frac{1+3q+3\beta}{1-3q+\beta} \cdot \frac{3-3q+\beta}{3+3q+\beta} \right) \\ &+ \frac{q}{\beta} \left(\arctan\left(\frac{3p}{3\beta-1}\right) + \arctan\left(\frac{3p}{3-\beta}\right) - \tan^{-1}\left(\frac{\sqrt{3}qk}{k^2\beta-1}\right) \right), \end{aligned}$$

where we understand the parameters k, β, p, q depending only on θ by

$$\begin{cases} k &= \frac{\sqrt{3}}{\tan\theta}, \\ \beta &= \sqrt{\frac{25+3k^2}{9+3k^2}}, \end{cases} \qquad \begin{cases} p &= \sqrt{\frac{(\beta+1)(5-3\beta)}{3}}, \\ q &= \sqrt{\frac{(\beta-1)(5+3\beta)}{3}}, \end{cases} \tag{1.3}$$

and arctan (*resp.* $\tan^{-1}$) *to be the principal branch* (*resp. the branch valued in* $(0, \pi]$) *of the arctangent function.*

In the extremal case of $\theta = \pi/2$, the above formula implies

Corollary B (Kanesaka [2])**.**

$$A\left(\frac{\pi}{2}\right) = \left(1 - \frac{2\sqrt{5}}{5}\right)\pi.$$

In the course of our proof of Theorem A, we first derive an explicit integral expression of $A(\theta)$ in §2. In §3, we perform the calculation of the integral and conclude the proof of Theorem A. In §4, we examine behaviors of some auxiliary quantities used in Theorem A and its proof, which help understanding convergence of individual terms of $A(\theta)$ in total to the value of $A(\frac{\pi}{2})$ in Corollary B and to $\lim_{\theta \to \frac{\pi}{3}} A(\theta) = 0$.

Before closing Introduction, we add one simple remark. In [3], M.Kontsevich and D.Zagier introduced the notion of *periods* as those complex numbers whose real and imaginary parts are integrals of algebraic functions over domains in $\mathbb{R}^n$ given by polynomial inequalities with algebraic coefficients, and proposed to check any special quantities to be periods in their sense. As for our $A(\theta)$, the following is a quick consequence of Theorem A.

Corollary C. *If* $\cot\theta$ *is a nonnegative algebraic number, then the real number $A(\theta)$ is a period in the sense of Kontsevich-Zagier* [3].

2. θ-acute region

In this section, we look into the boundary curve $\partial M(\theta)$ of the moduli region $M(\theta) \subset \mathcal{D}$. The following proposition generalizes [4] Remark 4.

Proposition 2.1. *Let k be the parameter as in (1.3). The points $re^{it} \in \partial M(\theta)$ are parametrized by the equation*

$$r = \begin{cases} \frac{1}{(1+k)^3}\left(2\cos\left(\frac{t-\pi}{3}\right) - \sqrt{4\cos^2\left(\frac{t-\pi}{3}\right) - (1-k^2)}\right)^3, & (\theta \neq \frac{2\pi}{3}), \\ \left(2\cos\left(\frac{t-\pi}{3}\right)\right)^{-3}, & (\theta = \frac{2\pi}{3}) \end{cases}$$

for $0 \le t < 2\pi$.

Proof. For any similarity class of triangles, we may choose a representative $\{0, 1, z\}$ with

$$z \in \mathcal{F} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0, |z| \le 1, |z-1| \le 1\}. \tag{2.2}$$

Noting that the maximum of the three angles of $\{0, 1, z\}$ is realized at the vertex $z \in \mathcal{F}$, we immediately see that the class of $\{0, 1, z\}$ belongs to $\partial M(\theta)$ if and only if $|z - \alpha| = |\alpha|$ with

$$\alpha := \frac{1}{2} + \frac{i}{2\tan\theta}. \tag{2.3}$$

Rewrite the condition $|z - \alpha| = |\alpha|$ in terms of $w := \frac{\rho^2 - \rho z}{z + \rho^2}$ by substituting $z = -\rho^2 \frac{w-1}{w+\rho}$. Then, using $\alpha + \bar{\alpha} = 1$, we find:

$$(1 + \bar{\alpha}\rho^2 + \alpha\bar{\rho}^2) w\bar{w} + 2\,\mathrm{Re}((\rho - 1)w) + (1 + \alpha\rho^2 + \overline{\alpha\rho^2}) = 0.$$

Now, put $w = \sqrt[3]{r} e^{\frac{it}{3}}$ so that $\mathrm{Re}(\rho^2 w) = -\sqrt[3]{r}\cos(\frac{t-\pi}{3})$. Then, since $\alpha\rho^2 + \overline{\alpha\rho^2} = -\frac{1}{2} - \frac{k}{2}$, $\bar{\alpha}\rho^2 + \alpha\bar{\rho}^2 = -\frac{1}{2} + \frac{k}{2}$, it yields a quadratic equation for $\sqrt[3]{r}$:

$$(1+k) r^{\frac{2}{3}} - 4\cos\left(\frac{t-\pi}{3}\right) r^{\frac{1}{3}} + (1-k) = 0.$$

Thus, we complete the proof of Proposition 2.1. □

Corollary 2.4. *Let $\frac{\pi}{3} \le \theta < \pi$ ($\theta \neq \frac{2\pi}{3}$) and let k be the parameter given in Theorem A. Then, the hyperbolic area $A(\theta)$ of $M(\theta)$ is given by*

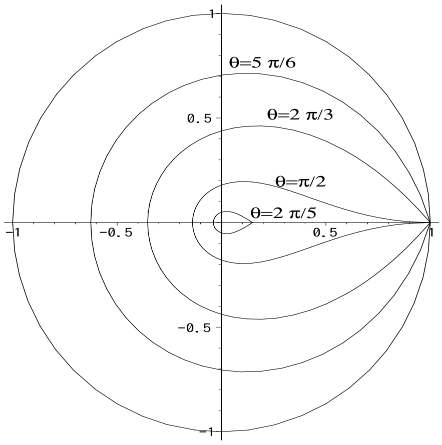
$$A(\theta) = 12 \int_0^{\frac{\pi}{3}} \left(\frac{(1+k)^6}{(1+k)^6 - \left(2\cos x - \sqrt{4\cos^2 x - (1-k^2)}\right)^6} - 1 \right) dx.$$

Proof. By the well known formula of hyperbolic geometry, we have

$$A(\theta) = \int_{M(\theta)} \frac{4}{(1-|z|^2)^2} dx dy = \int_0^{2\pi} \int_0^r \frac{2}{(1-r^2)^2} d(r^2) dt = \int_0^{2\pi} \frac{2r^2}{1-r^2} dt.$$

(Cf. e.g., [1] §5.3.) The corollary then immediately follows from Proposition 2.1 after substituting $x = \frac{t-\pi}{3}$. □

The following picture illustrates the loci $\{\sqrt{r}e^{it} \mid re^{it} \in \partial M(\theta)\}$ for $\theta = \frac{2\pi}{5}, \frac{\pi}{2}, \frac{2\pi}{3}$ and $\frac{5\pi}{6}$ respectively. Here, the polar scale is deformed from r to $\sqrt{r}$ to obtain illegible illustration of loci for small θ. Note that when $\theta = \frac{\pi}{3}$, the locus $\partial M(\frac{\pi}{3})$ degenerate to the point 0.



3. Proof of Theorem A

We shall evaluate the definite integral given in Corollary 2.4. As seen quickly below in Proposition 3.4, we may assume $\theta \neq \frac{2\pi}{3}$ without loss of generality. A brute force computation (by using Maple software) decomposes the integrand into three terms so that $A(\theta) = \int_0^{\frac{\pi}{3}} (A + B + C)dx$, where

(3.1)
$$A := -6, \quad B := -\frac{6(k^2+3)(3k^2+1)k}{S(\cos x)},$$

(3.2)
$$C := \frac{12\cos x(16\cos^2 x + k^2 - 1)(16\cos^2 x + 3k^2 - 3)\sqrt{4\cos^2 x - 1 + k^2}}{S(\cos x)}$$

with

$$S(X) = (4X^2 - 1)(16X^2 + 8X + 1 + 3k^2)(16X^2 - 8X + 1 + 3k^2).$$

We first check the convergence of the integral at $x = \frac{\pi}{3}$. It is not difficult to see that the Taylor expansion in $u := \frac{\pi}{3} - x$ reads:

$$A + B + C = \begin{cases} -6 - \frac{k}{\sqrt{3}u} + \frac{|k|}{\sqrt{3}u} + O(1), & (k \neq 0), \\ -6 + \frac{2}{\sqrt{2\sqrt{3}}}\frac{1}{\sqrt{u}} + O(1), & (k = 0). \end{cases} \quad (3.3)$$

Noting that $A(\theta)$ increases monotonously with θ, we immediately get

Proposition 3.4. *The area* $A(\theta) = \infty$ *for* $\theta > \frac{\pi}{2}$, *while it is finite for* $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$. □

In the following, we evaluate the case $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, i.e., $1 \geq k \geq 0$. We need to take care of annihilation of the divergence $\pm\frac{k}{\sqrt{3}u}$ from the terms B and C in (3.3). For the term B, let us look at the partial fraction decompositions $B = B_1 + B_2 + B_3$ with

$$B_1 = \frac{-2k}{4\cos^2 x - 1}, \tag{3.5}$$

$$B_2 := \frac{8k(1+\cos x)}{(1+4\cos x)^2 + 3k^2}, \tag{3.6}$$

$$B_3 := \frac{8k(1-\cos x)}{(1-4\cos x)^2 + 3k^2}. \tag{3.7}$$

Divergence of $\int B dx$ comes from the term B_1. In fact, using the formula $\int \frac{1}{2\cos x \mp 1} dx = \frac{1}{\sqrt{3}} \log(\frac{1+\sqrt{3}^{\pm 1}\tan\frac{x}{2}}{1-\sqrt{3}^{\pm 1}\tan\frac{x}{2}})$, we see that

$$\int_0^{\frac{\pi}{3}} B_1 dx = \lim_{x \to \frac{\pi}{3}} \frac{k}{\sqrt{3}} \left(\log(\frac{4\cdot 3\cdot 1}{3\cdot 2\cdot 2}) + \log\left(1 - \sqrt{3}\tan\frac{x}{2}\right)\right). \tag{3.8}$$

To evaluate the term C, let us substitute $\sin x = \frac{\sqrt{3+k^2}}{2}\frac{t^2-1}{t^2+1}$ so that

$$t = \frac{\sqrt{3+k^2} + 2\sin x}{\sqrt{3+k^2 - 4\sin^2 x}} \tag{3.9}$$

and that $0 \leq x \leq \frac{\pi}{3}$ corresponds to $1 \leq t \leq \frac{\sqrt{3+k^2}+\sqrt{3}}{k}$. Noting that $\sqrt{4\cos^2 x - 1 + k^2} = \sqrt{3+k^2}\frac{2t}{t^2+1}$, $\cos x dx = \frac{2t\sqrt{3+k^2}}{(t^2+1)^2}dt$, we obtain the decomposition $C dx = (C_1 + C_2 + C_3)dt$ with

$$C_1 = -8(t^2+1)\frac{3(t^4+18t^2+1) + (t^4-14t^2+1)k^2}{3(t^4+18t^2+1)^2 + (t^4-14t^2+1)^2 k^2}, \tag{3.10}$$

$$C_2 = \frac{-4k^2(t^2+1)}{k^2t^4 - 12t^2 - 2k^2t^2 + k^2}, \tag{3.11}$$

$$C_3 = \frac{12}{t^2+1}. \tag{3.12}$$

Divergence from $\int C dx$ comes from the term

$$\int_1^{\frac{\sqrt{3+k^2}+\sqrt{3}}{k}} C_2\, dt = \frac{k}{\sqrt{3}}\left[\log\left|\frac{t^2+\frac{2\sqrt{3}}{k}t-1}{t^2-\frac{2\sqrt{3}}{k}t-1}\right|\right]_1^{\frac{\sqrt{3+k^2}+\sqrt{3}}{k}} \tag{3.13}$$

$$=\frac{k}{\sqrt{3}}\left[\log\frac{(\sqrt{3+k^2}+k-\sqrt{3})\cdot 2\sqrt{3}\cdot 2(\sqrt{3}+\sqrt{3+k^2})}{2\sqrt{3+k^2}\cdot(\sqrt{3}+k-\sqrt{3+k^2})\cdot(\sqrt{3}+k+\sqrt{3+k^2})}\right]$$

$$+\lim_{x\to\frac{\pi}{3}}\frac{k}{\sqrt{3}}\log\left(\frac{\sqrt{3+k^2}+\sqrt{3}-k}{\sqrt{3+k^2}+\sqrt{3}-kt}\right).$$

The sum of (3.8) and the last term of (3.13) can be computed by l'Hôpital's rule as

$$\frac{k}{\sqrt{3}}\log\left(\frac{\sqrt{3+k^2}+\sqrt{3}-k}{\sqrt{3+k^2}+\sqrt{3}}\lim_{x\to\frac{\pi}{3}}\frac{1-\sqrt{3}\tan\frac{x}{2}}{1-\frac{k}{\sqrt{3+k^2}+\sqrt{3}}\cdot\frac{\sqrt{3+k^2}+2\sin x}{\sqrt{3+k^2-4\sin^2 x}}}\right)$$

$$=\frac{k}{\sqrt{3}}\log\left(\frac{2k^2(\sqrt{3+k^2}+\sqrt{3}-k)}{\sqrt{3}\sqrt{k^2+3}(\sqrt{3+k^2}+\sqrt{3})}\right).$$

Putting this together with the rest term of (3.13), we obtain

$$\int_0^{\frac{\pi}{3}} B_1\, dx + \int_1^{\frac{\sqrt{3+k^2}+\sqrt{3}}{k}} C_2\, dt = \frac{k}{\sqrt{3}}\log\left(\frac{4k^2}{k^2+3}\right). \tag{3.14}$$

We shall next compute $\int (B_2+B_3)dx$. The standard substitutions

$$\begin{cases} t=\tan\frac{x}{2} & \text{for } B_2 dx,\\ t=\cot\frac{x}{2} & \text{for } B_3 dx\end{cases} \tag{3.15}$$

transform it as :

(3.16)

$$\int_0^{\frac{\pi}{3}}(B_2+B_3)dx = \left(\int_0^{\frac{\sqrt{3}}{3}}+\int_{\sqrt{3}}^{\infty}\right)\frac{32k\, dt}{(9+3k^2)t^4+(6k^2-30)t^2+(25+3k^2)}.$$

Now, we introduce the quantities

$$\begin{cases}\alpha := \frac{k^2-5}{k^2+3},\\ \beta := \sqrt{\frac{3k^2+25}{3k^2+9}},\end{cases}\qquad \begin{cases} p := \sqrt{\frac{(\beta+1)(5-3\beta)}{3}},\\ q := \sqrt{\frac{(\beta-1)(5+3\beta)}{3}},\end{cases} \tag{3.17}$$

which satisfy the following relations:

$$\beta+\alpha = \frac{3}{2}p^2, \quad \beta-\alpha=\frac{3}{2}q^2. \tag{3.18}$$

Then we find an indefinite integral for (3.16) can be performed as

$$\int \frac{32k\ dt}{(3k^2+9)(t^2+\sqrt{2(\beta-\alpha)}t+\beta)(t^2-\sqrt{2(\beta-\alpha)}t+\beta)}$$
$$= \frac{p}{2\beta}\log\frac{t^2+\sqrt{3}qt+\beta}{t^2-\sqrt{3}qt+\beta}$$
$$+\frac{q}{\beta}\left\{\arctan\left(\frac{2t}{\sqrt{3}p}+\frac{q}{p}\right)+\arctan\left(\frac{2t}{\sqrt{3}p}-\frac{q}{p}\right)\right\}.$$

From this it follows that

$$\int_0^{\frac{\pi}{3}}(B_2+B_3)dx = \frac{p}{2\beta}\log\left(\frac{1+3q+3\beta}{1-3q+\beta}\cdot\frac{3-3q+\beta}{3+3q+\beta}\right) + \frac{q}{\beta}\left(\arctan\left(\frac{3p}{3\beta-1}\right)+\arctan\left(\frac{3p}{3-\beta}\right)\right). \tag{3.19}$$

As for the integral $\int C_1 dt$, observe that the integrand can be transformed in the simpler variable $s=\frac{1}{2}(t-\frac{1}{t})$: thus we obtain

(3.20)

$$\int_1^{\frac{\sqrt{3+k^2}+\sqrt{3}}{k}} C_1\,dt = -4\int_0^{\frac{\sqrt{3}}{k}}\frac{3(s^2+5)+(s^2-3)k^2}{3(s^2+5)^2+(s^2-3)^2k^2}ds$$
$$= -4\int_0^{\frac{\sqrt{3}}{k}}\frac{s^2-3\alpha}{(s^2+\sqrt{6(\alpha+\beta)}s+3\beta)(s^2-\sqrt{6(\alpha+\beta)}s+3\beta)}ds$$
$$= -\frac{p}{2\beta}\log\left(\frac{1-\sqrt{3}pk+\beta k^2}{1+\sqrt{3}pk+\beta k^2}\right)$$
$$-\frac{q}{\beta}\left[\arctan\left(\frac{2s}{3q}+\frac{p}{q}\right)+\arctan\left(\frac{2s}{3q}-\frac{p}{q}\right)\right]_0^{\frac{\sqrt{3}}{k}}$$
$$= -\frac{p}{2\beta}\log\left(\frac{1-\sqrt{3}pk+\beta k^2}{1+\sqrt{3}pk+\beta k^2}\right)-\frac{q}{\beta}\tan^{-1}\left(\frac{\sqrt{3}qk}{k^2\beta-1}\right).$$

Finally, the remaining integral can be given by

$$\int_0^{\frac{\pi}{3}}(-6)dx+\int_1^{\frac{\sqrt{3+k^2}+\sqrt{3}}{k}}\frac{12dt}{t^2+1} = -2\pi+12\arctan\left(\frac{\sqrt{3+k^2}+\sqrt{3}}{k}\right)-3\pi.$$

But since $\frac{\sqrt{3+k^2}+\sqrt{3}}{k}=\tan(\frac{\pi}{4}+\frac{\theta}{2})$, it equals to $6\theta-2\pi$. This, together with (3.14), (3.19), (3.20), concludes the proof of Theorem A. □

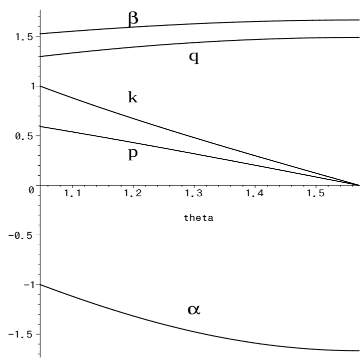
4. Behaviors of auxiliary quantities and Corollaries B and C

In this section, we shall closely examine respective terms of our explicit formula of $A(\theta)$ in Theorem A:

$$\begin{aligned} A(\theta) =&(6\theta-2\pi)+\frac{k}{\sqrt{3}}\log\frac{4k^2}{k^2+3} \\ &+\frac{p}{2\beta}\log\left(\frac{1+\sqrt{3}pk+\beta k^2}{1-\sqrt{3}pk+\beta k^2}\cdot\frac{1+3q+3\beta}{1-3q+\beta}\cdot\frac{3-3q+\beta}{3+3q+\beta}\right) \\ &+\frac{q}{\beta}\left(\arctan(\frac{3p}{3\beta-1})+\arctan(\frac{3p}{3-\beta})-\tan^{-1}(\frac{\sqrt{3}qk}{k^2\beta-1})\right). \end{aligned}$$

First, we shall look at the quantities k, α, β, p, q introduced in (1.3) and (3.17) with respect to the parameter $\theta\in[\frac{\pi}{3},\frac{\pi}{2}]$. In fact, by simple estimation, the following table and graph can be derived, where approximately $\frac{\sqrt{21}}{3}\approx 1.5275$, $\frac{\sqrt{2\sqrt{21}-6}}{3}\approx 0.593$, $\frac{\sqrt{2\sqrt{21}+6}}{3}\approx 1.298$.

θ	$\frac{\pi}{3}$	$\dots$	$\frac{\pi}{2}$
k	1	$\searrow$	0
α	-1	$\searrow$	$-\frac{5}{3}$
β	$\frac{\sqrt{21}}{3}$	$\nearrow$	$\frac{5}{3}$
p	$\frac{\sqrt{2\sqrt{21}-6}}{3}$	$\searrow$	0
q	$\frac{\sqrt{2\sqrt{21}+6}}{3}$	$\nearrow$	$\frac{2}{3}\sqrt{5}$



These quantities are also related by (3.18) and

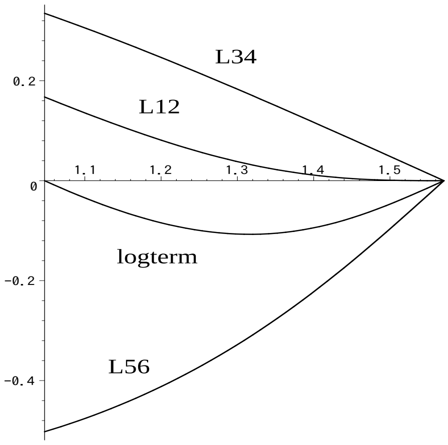
$$\alpha=1-\frac{8}{3}\sin^2\theta,\quad \beta=\frac{1}{3}\sqrt{9+16\sin^2\theta}, \tag{4.1}$$

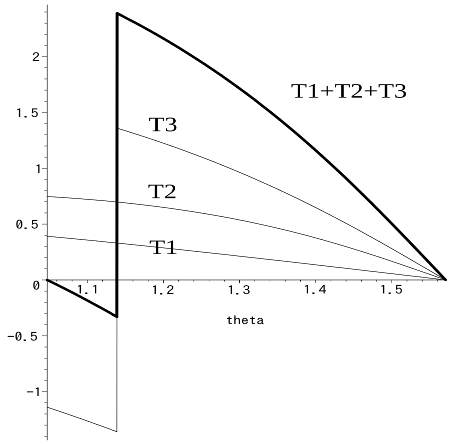
$$pq=\frac{8}{9}\sin 2\theta. \tag{4.2}$$

Let us now examine behaviors of the main logarithmic term and the arctangent term of Theorem A. Set

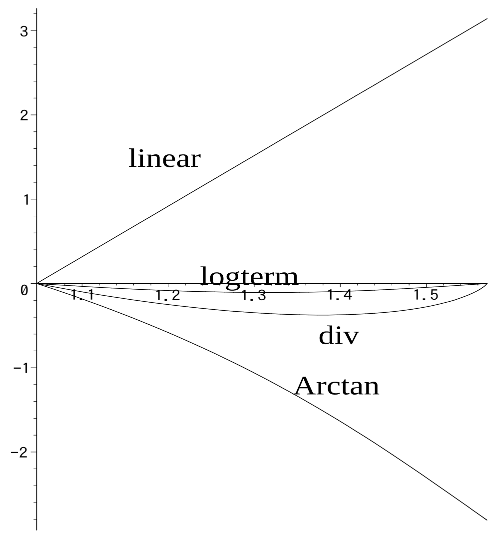
$$\begin{cases} L12 &:= \frac{p}{2\beta}\log\left(\frac{1+\sqrt{3}pk+\beta k^2}{1-\sqrt{3}pk+\beta k^2}\right), \\ L34 &:= \frac{p}{2\beta}\log\left(\frac{1+3q+3\beta}{1-3q+\beta}\right), \\ L56 &:= \frac{p}{2\beta}\log\left(\frac{3-3q+\beta}{3+3q+\beta}\right), \end{cases}\qquad \begin{cases} T1 &:= \frac{q}{\beta}\arctan(\frac{3p}{3\beta-1}), \\ T2 &:= \frac{q}{\beta}\arctan(\frac{3p}{3-\beta}), \\ T3 &:= -\frac{q}{\beta}\arctan(\frac{\sqrt{3}qk}{k^2\beta-1}). \end{cases}$$

Illustration of these quantities together with 'logterm' $= L12+L34+L56$ and $T1+T2+T3$ are given as follows:





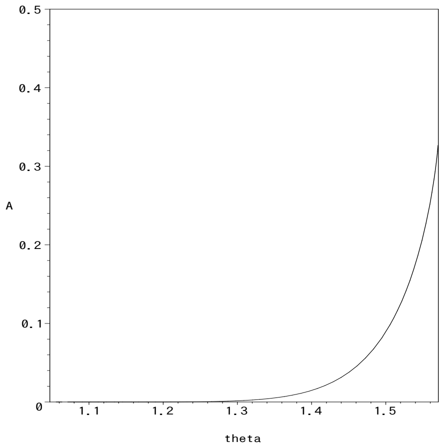
The term T_3 has a jump when $k^2\beta = 1$, i.e., approximately at $\theta = 1.139$. In Theorem A, we request the branch $\tan^{-1}$ to correct T_3 to $\tilde{T}_3$ so that the sum 'Arctan'$= T_1 + T_2 + \tilde{T}_3$ has values continuously ranging from 0 down to $-\frac{2\sqrt{5}}{5}\pi$. (N.B. The term $\tilde{T}_3$ itself has values contained in $[-\frac{2\sqrt{5}}{5}\pi, -1)$.) This latter value is nothing but $-\frac{q}{\beta}\pi$ at $\theta = \frac{\pi}{2}$. Besides the above 'logterm' and 'Arctan' the following picture collects the 'linear' term $6\theta - 2\pi$ and the 'div' term (which cancels divergence from B_1 (3.5) and that from C_2 (3.11) in their correct balance (3.14) as to be) $\frac{k}{\sqrt{3}} \log \frac{4k^2}{k^2+3}$:



We conclude that the behavior of the total of these four terms

$$A(\theta) = \text{'linear'} + \text{'div'} + \text{'logterm'} + \text{'Arctan'}$$

is illustrated as in the following graph, whose curve starts from 0 at $\theta = \frac{\pi}{3}$ and terminates with the value $(1 - \frac{2\sqrt{5}}{5})\pi$ at $\theta = \frac{\pi}{2}$.



Finally, let us mention a few words on Corollary C. According to the definition of periods due to Kontsevich-Zagier [3], it is immediate to see that

$$\pi = \int_{x^2+y^2\leq 1} dxdy, \quad \log(\alpha) = \int_1^\alpha \frac{1}{x}dx, \quad \arctan(\alpha) = \int_0^\alpha \frac{1}{x^2+1}dx$$

are periods in their sense for any positive algebraic number $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$. Thus, Corollary C follows from Theorem A and the above estimation of involved quantities.

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