Math. J. Okayama Univ. **59** (2017), 131–140

HIGHER-DIMENSIONAL ABSOLUTE VERSIONS OF SYMMETRIC, FROBENIUS, AND QUASI-FROBENIUS ALGEBRAS

Mitsuyasu Hashimoto

Abstract. In this paper, we define and discuss higher-dimensional and absolute versions of symmetric, Frobenius, and quasi-Frobenius algebras. In particular, we compare these with the relative notions defined by Scheja and Storch. We also prove the validity of codimension two-argument for modules over a coherent sheaf of algebras with a 2-canonical module, generalizing a result of the author.

1. Introduction

(1.1) Let $(R, \mathfrak{m})$ be a semilocal Noetherian commutative ring, and Λ a module-finite R-algebra. In [6], we defined the canonical module K_Λ of Λ. The purpose of this paper is two fold, each of which is deeply related to K_Λ.

(1.2) In the first part, we define and discuss higher-dimensional and absolute notions of symmetric, Frobenius, and quasi-Frobenius algebras and their non-Cohen–Macaulay versions. In commutative algebra, the non-Cohen–Macaulay version of Gorenstein ring is known as quasi-Gorenstein rings. What we discuss here is a non-commutative version of such rings. Scheja and Storch [7] discussed a relative notion, and our definition is absolute in the sense that it depends only on Λ and is independent of the choice of R. If R is local, our quasi-Frobenius property agrees with Gorensteinness discussed by Goto and Nishida [1], see Proposition 3.6 and Corollary 3.7.

(1.3) In the second part, we show that the codimension-two argument using the existence of 2-canonical modules in [4] is also still valid in non-commutative settings. For the definition of an n-canonical module, see (2.8). Codimension-two argument, which states (roughly speaking) that removing a closed subset of codimension two or more does not change the category of coherent sheaves which satisfy Serre's (S'_2) condition, is sometimes used in algebraic geometry, commutative algebra and invariant theory. For example, information on the canonical sheaf and the class group is retained when we remove the singular locus of a normal variety over an algebraically closed field, and then these objects are respectively grasped as the top exterior

Mathematics Subject Classification. Primary 16E65; Secondary 14A15.

Key words and phrases. canonical module, symmetric algebra, Frobenius algebra, quasi-Frobenius algebra, n-canonical module.

power of the cotangent bundle and the Picard group of a smooth variety. In [4], almost principal bundles are studied. They are principal bundles after removing closed subsets of codimension two or more.

We prove the following. Let X be a locally Noetherian scheme, U an open subset of X such that $\operatorname{codim}_X(X \setminus U) \geq 2$. Let $i : U \to X$ be the inclusion. Let Λ be a coherent $\mathcal{O}_X$-algebra. If X possesses a 2-canonical module ω, then the inverse image i^* induces the equivalence between the category of coherent right Λ-modules which satisfy the (S_2') condition and the category of coherent right $i^*\Lambda$-modules which satisfy the (S_2') condition. The quasi-inverse is given by the direct image i_*. What was proved in [4] was the case that $\Lambda = \mathcal{O}_X$. If, moreover, $\omega = \mathcal{O}_X$ (that is to say, X satisfy the (S_2) and (G_1) condition), then the assertion has been well-known, see [3].

(1.4) 2-canonical modules are ubiquitous in algebraic geometry. If $\mathbb{I}$ is a dualizing complex of a Noetherian scheme X, then the lowest non-vanishing cohomology group of $\mathbb{I}$ is semicanonical. A rank-one reflexive sheaf over a normal variety is 2-canonical.

(1.5) Section 2 is for preliminaries. Section 3 is devoted to the discussion of the first theme mentioned in the paragraph (1.2), while Section 4 is for the second theme mentioned in (1.3).

(1.6) Acknowledgments: Special thanks are due to Professor Osamu Iyama for valuable advice and discussion.

The essential part of this paper has first appeared as [5, sections 9–10]. When it is published as [6], they have been removed after the requirement to shorten the paper (also, the title has been changed slightly). Here we revive them as an independent paper.

2. Preliminaries

(2.1) Throughout this paper, R denotes a Noetherian commutative ring. For a module-finite R-algebra Λ, a Λ-module means a left Λ-module. Λ^{op} denotes the opposite algebra of Λ, and thus a Λ^{op}-module is identified with a right Λ-module. A Λ-bimodule means a $\Lambda \otimes_R \Lambda^{\mathrm{op}}$-module. The category of finite Λ-modules is denoted by $\Lambda \operatorname{mod}$. The category $\Lambda^{\mathrm{op}} \operatorname{mod}$ is also denoted by $\operatorname{mod} \Lambda$.

(2.2) Let $(R, \mathfrak{m})$ be semilocal and Λ be a module-finite R-algebra. For an R-module M, the $\mathfrak{m}$-adic completion of M is denoted by $\hat{M}$. For a finite Λ-module M, by $\dim M$ or $\dim_\Lambda M$ we mean $\dim_R M$, which is independent of the choice of R. By $\operatorname{depth} M$ or $\operatorname{depth}_\Lambda M$ we mean $\operatorname{depth}_R(\mathfrak{m}, M)$, which is independent of R. We say that M is globally Cohen–Macaulay (GCM for short) if $\dim M = \operatorname{depth} M$. We say that M is globally maximal Cohen–Macaulay (GMCM for short) if $\dim \Lambda = \operatorname{depth} M$. If R happens to be local, then M is GCM (resp. GMCM) if and only if M is Cohen–Macaulay (resp. maximal Cohen–Macaulay) as an R-module.

(2.3) For $M \in \Lambda \operatorname{mod}$, we say that M satisfies $(S'_n)^{\Lambda,R}$, $(S'_n)^R$ or (S'_n) if $\operatorname{depth}_{R_P} M_P \geq \min(n, \operatorname{ht}_R P)$ for every $P \in \operatorname{Spec} R$ (this notion depends on R).

(2.4) Let X be a locally Noetherian scheme and Λ a coherent $\mathcal{O}_X$-algebra. For a coherent Λ-module $\mathcal{M}$, we say that $\mathcal{M}$ satisfies (S'_n) or $(S'_n)^{\Lambda,X}$, or sometimes $\mathcal{M} \in (S'_n)^{\Lambda,X}$, if $\operatorname{depth}_{\mathcal{O}_{X,x}} \mathcal{M}_x \geq \min(n, \dim \mathcal{O}_{X,x})$ for every $x \in X$.

(2.5) Assume that $(R, \mathfrak{m})$ is complete semilocal, and $\Lambda \neq 0$ a module-finite R-algebra. Let $\mathbb{I}$ be a normalized dualizing complex of R. The lowest non-vanishing cohomology group $\operatorname{Ext}_R^{-s}(\Lambda, \mathbb{I})$ ($\operatorname{Ext}_R^i(\Lambda, \mathbb{I}) = 0$ for $i < -s$) is denoted by K_Λ, and is called the *canonical module* of Λ. If $\Lambda = 0$, then we define that $K_\Lambda = 0$. For basics on the canonical modules, we refer the reader to [6]. Note that K_Λ depends only on Λ, and is independent of R.

(2.6) Assume that $(R, \mathfrak{m})$ is semilocal which may not be complete. We say that a finitely generated Λ-bimodule K is a *canonical module* of Λ if $\hat{K}$ is isomorphic to the canonical module $K_{\hat{\Lambda}}$ as a $\hat{\Lambda}$-bimodule. It is unique up to isomorphisms, and denoted by K_Λ. We say that $K \in \operatorname{mod} \Lambda$ is a right (resp. left) canonical module of Λ if $\hat{K}$ is isomorphic to $K_{\hat{\Lambda}}$ in $\operatorname{mod} \hat{\Lambda}$ (resp. $\hat{\Lambda} \operatorname{mod}$). If K_Λ exists, then K is a right canonical module if and only if $K \cong K_\Lambda$ in $\operatorname{mod} \Lambda$.

(2.7) We say that ω is an R-semicanonical right Λ-module if for any $P \in \operatorname{Spec} R$, $R_P \otimes_R \omega$ is the right canonical module $R_P \otimes_R \Lambda$ for any $P \in \operatorname{supp}_R \omega$.

(2.8) Let $C \in \operatorname{mod} \Lambda$. We say that C is an n-canonical right Λ-module over R if $C \in (S'_n)^R$, and for each $P \in \operatorname{Spec} R$ with $\operatorname{ht} P < n$, we have that C_P is an R_P-semicanonical right Λ_P-module.

3. Symmetric and Frobenius algebras

(3.1) Let $(R, \mathfrak{m})$ be a Noetherian semilocal ring, and Λ a module-finite R-algebra. Let K_Λ denote the canonical module of Λ, see [6].

We say that Λ is *quasi-symmetric* if Λ is the canonical module of Λ. That is, $\Lambda \cong K_\Lambda$ as Λ-bimodules. It is called *symmetric* if it is quasi-symmetric and GCM. Note that Λ is quasi-symmetric (resp. symmetric) if and only if $\hat{\Lambda}$ is so. Note also that quasi-symmetric and symmetric are absolute notion, and is independent of the choice of R in the sense that the definition does not change when we replace R by the center of Λ, because K_Λ is independent of the choice of R.

(3.2) For (non-semilocal) Noetherian ring R, we say that Λ is locally quasi-symmetric (resp. locally symmetric) over R if for any $P \in \operatorname{Spec} R$, Λ_P is a quasi-symmetric (resp. symmetric) R_P-algebra. This is equivalent to say that for any maximal ideal $\mathfrak{m}$ of R, $\Lambda_\mathfrak{m}$ is quasi-symmetric (resp. symmetric), see [6, (7.6)].

In the case that $(R, \mathfrak{m})$ is semilocal, Λ is locally quasi-symmetric (resp. locally symmetric) over R if it is quasi-symmetric (resp. symmetric), but the converse is not true in general.

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian semilocal ring, and Λ a module-finite R-algebra. Then the following are equivalent.*

1 *Λ_Λ is the right canonical module of Λ.*
2 *${}_\Lambda\Lambda$ is the left canonical module of Λ.*

Proof. We may assume that R is complete. Then replacing R by a Noether normalization of $R/\operatorname{ann}_R \Lambda$, we may assume that R is regular and Λ is a faithful R-module.

We prove **1**$\Rightarrow$**2**. By [6, Lemma 5.10], K_Λ satisfies $(S_2')^R$. By assumption, Λ_Λ satisfies $(S_2')^R$. As R is regular and $\dim R = \dim \Lambda$, $K_\Lambda = \Lambda^* = \operatorname{Hom}_R(\Lambda, R)$. So we get an R-linear map

$$\varphi : \Lambda \otimes_R \Lambda \to R$$

such that $\varphi(ab \otimes c) = \varphi(a \otimes bc)$ and that the induced map $h : \Lambda \to \Lambda^*$ given by $h(a)(c) = \varphi(a \otimes c)$ is an isomorphism (in $\operatorname{mod} \Lambda$). Now φ induces a homomorphism $h' : \Lambda \to \Lambda^*$ in $\Lambda \operatorname{mod}$ given by $h'(c)(a) = \varphi(a \otimes c)$. To verify that this is an isomorphism, as Λ and Λ^* are reflexive R-modules, we may localize at a prime P of R of height at most one, and then take a completion, and hence we may further assume that $\dim R \le 1$. Then Λ is a finite free R-module, and the matrices of h and h' are transpose each other. As the matrix of h is invertible, so is that of h', and h' is an isomorphism.

2$\Rightarrow$**1** follows from **1**$\Rightarrow$**2**, considering the opposite ring. $\square$

Definition 1. Let $(R, \mathfrak{m})$ be semilocal. We say that Λ is a *pseudo-Frobenius R-algebra* if the equivalent conditions of Lemma 3.3 are satisfied. If Λ is GCM in addition, then it is called a *Frobenius R-algebra*. Note that these definitions are independent of the choice of R. Moreover, Λ is pseudo-Frobenius (resp. Frobenius) if and only if $\hat{\Lambda}$ is so. For a general R, we say that Λ is locally pseudo-Frobenius (resp. locally Frobenius) over R if Λ_P is pseudo-Frobenius (resp. Frobenius) for $P \in \operatorname{Spec} R$.

Lemma 3.4. *Let $(R, \mathfrak{m})$ be semilocal. Then the following are equivalent.*

1 $(K_{\hat{\Lambda}})_{\hat{\Lambda}}$ *is projective in* $\operatorname{mod} \hat{\Lambda}$.
2 ${}_{\hat{\Lambda}}(K_{\hat{\Lambda}})$ *is projective in* $\hat{\Lambda} \operatorname{mod}$,

where $\hat{?}$ *denotes the* $\mathfrak{m}$*-adic completion.*

Proof. We may assume that $(R, \mathfrak{m}, k)$ is complete regular local and Λ is a faithful R-module. Let $\bar{?}$ denote the functor $k \otimes_R ?$. Then $\bar{\Lambda}$ is a finite dimensional k-algebra. So $\operatorname{mod} \bar{\Lambda}$ and $\bar{\Lambda} \operatorname{mod}$ have the same number of simple modules, say n. An indecomposable projective module in $\operatorname{mod} \Lambda$ is nothing but the projective cover of a simple module in $\operatorname{mod} \bar{\Lambda}$. So $\operatorname{mod} \Lambda$ and $\Lambda \operatorname{mod}$ have n indecomposable projectives. Now $\operatorname{Hom}_R(?, R)$ is an equivalence between $\operatorname{add}(K_\Lambda)_\Lambda$ and $\operatorname{add} {}_\Lambda \Lambda$. It is also an equivalence between $\operatorname{add} {}_\Lambda(K_\Lambda)$ and $\operatorname{add} \Lambda_\Lambda$. So both $\operatorname{add}(K_\Lambda)_\Lambda$ and $\operatorname{add} {}_\Lambda(K_\Lambda)$ also have n indecomposables. So **1** is equivalent to $\operatorname{add}(K_\Lambda)_\Lambda = \operatorname{add} \Lambda_\Lambda$. **2** is equivalent to $\operatorname{add} {}_\Lambda(K_\Lambda) = \operatorname{add} {}_\Lambda \Lambda$. So **1**$\Leftrightarrow$**2** is proved simply applying the duality $\operatorname{Hom}_R(?, R)$. $\square$

(3.5) Let $(R, \mathfrak{m})$ be semilocal. If the equivalent conditions in Lemma 3.4 are satisfied, then we say that Λ is *pseudo-quasi-Frobenius.* If it is GCM in addition, then we say that it is *quasi-Frobenius.* These definitions are independent of the choice of R. Note that Λ is pseudo-quasi-Frobenius (resp. quasi-Frobenius) if and only if $\hat{\Lambda}$ is so.

Proposition 3.6. *Let $(R, \mathfrak{m})$ be semilocal. Then the following are equivalent.*

1 Λ *is quasi-Frobenius.*
2 Λ *is GCM, and* $\dim \Lambda = \operatorname{idim} {}_\Lambda \Lambda$*, where* idim *denotes the injective dimension.*
3 Λ *is GCM, and* $\dim \Lambda = \operatorname{idim} \Lambda_\Lambda$.

Proof. **1**$\Rightarrow$**2**. By definition, Λ is GCM. To prove that $\dim \Lambda = \operatorname{idim} {}_\Lambda \Lambda$, we may assume that R is local. Then by [1, (3.5)], we may assume that R is complete. Replacing R by the Noetherian normalization of $R / \operatorname{ann}_R \Lambda$, we may assume that R is a complete regular local ring of dimension d, and Λ its maximal Cohen–Macaulay (that is, finite free) module. As $\operatorname{add} {}_\Lambda \Lambda = \operatorname{add} {}_\Lambda(K_\Lambda)$ by the proof of Lemma 3.4, it suffices to prove $\operatorname{idim} {}_\Lambda(K_\Lambda) = d$. Let $\mathbb{I}_R$ be the minimal injective resolution of the R-module R. Then $\mathbb{J} = \operatorname{Hom}_R(\Lambda, \mathbb{I}_R)$ is an injective resolution of $K_\Lambda = \operatorname{Hom}_R(\Lambda, R)$ as a left Λ-module. As the length of $\mathbb{J}$ is d and

$$\operatorname{Ext}^d_\Lambda(\Lambda/\mathfrak{m}\Lambda, K_\Lambda) \cong \operatorname{Ext}^d_R(\Lambda/\mathfrak{m}\Lambda, R) \neq 0,$$

we have that $\operatorname{idim} {}_\Lambda(K_\Lambda) = d$.

2⇒1. We may assume that R is complete regular local and Λ is maximal Cohen–Macaulay. By [1, (3.6)], we may further assume that R is a field. Then ${}_\Lambda\Lambda$ is injective. So $(K_\Lambda)_\Lambda = \operatorname{Hom}_R(\Lambda, R)$ is projective, and Λ is quasi-Frobenius, see [8, (IV.3.7)].

1⇔3 is proved similarly. □

Corollary 3.7. *Let R be arbitrary. Then the following are equivalent.*

1 *For any $P \in \operatorname{Spec} R$, Λ_P is quasi-Frobenius.*

2 *For any maximal ideal $\mathfrak{m}$ of R, $\Lambda_\mathfrak{m}$ is quasi-Frobenius.*

3 *Λ is a Gorenstein R-algebra in the sense that Λ is a Cohen–Macaulay R-module, and $\operatorname{idim}_{\Lambda_P} {}_{\Lambda_P}\Lambda_P = \dim \Lambda_P$ for any $P \in \operatorname{Spec} R$.*

Proof. **1⇒2** is trivial.

2⇒3. By Proposition 3.6, we have $\operatorname{idim} {}_{\Lambda_\mathfrak{m}}\Lambda_\mathfrak{m} = \dim \Lambda_\mathfrak{m}$ for each $\mathfrak{m}$. Then by [1, (4.7)], Λ is a Gorenstein R-algebra.

3⇒1 follows from Proposition 3.6. □

(3.8) Let R be arbitrary. We say that Λ is a *quasi-Gorenstein* R-algebra if Λ_P is pseudo-quasi-Frobenius for each $P \in \operatorname{Spec} R$.

Definition 2 (Scheja–Storch [7])**.** Let R be general. We say that Λ is symmetric (resp. Frobenius) relative to R if Λ is R-projective, and $\Lambda^* := \operatorname{Hom}_R(\Lambda, R)$ is isomorphic to Λ as a Λ-bimodule (resp. as a right Λ-module). It is called quasi-Frobenius relative to R if the right Λ-module Λ^* is projective.

Lemma 3.9. *Let $(R, \mathfrak{m})$ be local.*

1 *If $\dim \Lambda = \dim R$, R is quasi-Gorenstein, and $\Lambda^* \cong \Lambda$ as Λ-bimodules (resp. $\Lambda^* \cong \Lambda$ as right Λ-modules, Λ^* is projective as a right Λ-module), then Λ is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius).*

2 *Assume that R is Gorenstein. If Λ is symmetric (resp. Frobenius, quasi-Frobenius) relative to R, then Λ is symmetric (resp. Frobenius, quasi-Frobenius).*

3 *If Λ is nonzero and R-projective, then Λ is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius) if and only if R is quasi-Gorenstein and Λ is symmetric (resp. Frobenius, quasi-Frobenius) relative to R.*

4 *If Λ is nonzero and R-projective, then Λ is symmetric (resp. Frobenius, quasi-Frobenius) if and only if R is Gorenstein and Λ is symmetric (resp. Frobenius, quasi-Frobenius) relative to R.*

Proof. We can take the completion, and we may assume that R is complete local.

1. Let $d = \dim \Lambda = \dim R$, and let $\mathbb{I}$ be the normalized dualizing complex (see [6, (5.2)]) of R. Then

$$K_\Lambda = \operatorname{Ext}_R^{-d}(\Lambda, \mathbb{I}) \cong \operatorname{Hom}_R(\Lambda, H^{-d}(\mathbb{I})) \cong \operatorname{Hom}(\Lambda, K_R) \cong \operatorname{Hom}(\Lambda, R) = \Lambda^*$$

as Λ-bimodules, and the result follows.

2. We may assume that Λ is nonzero. As R is Cohen–Macaulay and Λ is a finite projective R-module, Λ is a maximal Cohen–Macaulay R-module. By **1**, the result follows.

3. The 'if' part follows from **1**. We prove the 'only if' part. As Λ is R-projective and nonzero, $\dim \Lambda = \dim R$. As Λ is R-finite free, $K_\Lambda \cong \operatorname{Hom}_R(\Lambda, K_R) \cong \Lambda^* \otimes_R K_R$. As K_Λ is R-free and $\Lambda^* \otimes_R K_R$ is nonzero and is isomorphic to a direct sum of copies of K_R, we have that K_R is R-projective, and hence R is quasi-Gorenstein, and $K_R \cong R$. Hence $K_\Lambda \cong \Lambda^*$, and the result follows.

4 follows from **3** easily. $\square$

(3.10) Let $(R, \mathfrak{m})$ be semilocal. Let a finite group G act on Λ by R-algebra automorphisms. Let $\Omega = \Lambda * G$, the twisted group algebra. That is, $\Omega = \Lambda \otimes_R RG = \bigoplus_{g \in G} \Lambda g$ as an R-module, and the product of Ω is given by $(ag)(a'g') = (a(ga'))(gg')$ for $a, a' \in \Lambda$ and $g, g' \in G$. This makes Ω a module-finite R-algebra.

(3.11) We simply call an RG-module a G-module. We say that M is a (G, Λ)-module if M is a G-module, Λ-module, the R-module structures coming from that of the G-module structure and the Λ-module structure agree, and $g(am) = (ga)(gm)$ for $g \in G$, $a \in \Lambda$, and $m \in M$. A (G, Λ)-module and an Ω-module are one and the same thing.

(3.12) By the action $((a \otimes a')g)a_1 = a(ga_1)a'$, we have that Λ is a $(\Lambda \otimes \Lambda^{\mathrm{op}}) * G$-module in a natural way. So it is an Ω-module by the action $(ag)a_1 = a(ga_1)$. It is also a right Ω-module by the action $a_1(ag) = g^{-1}(a_1 a)$. If the action of G on Λ is trivial, then these actions make an Ω-bimodule.

(3.13) Given an Ω-module M and an RG-module V, $M \otimes_R V$ is an Ω-module by $(ag)(m \otimes v) = (ag)m \otimes gv$. $\operatorname{Hom}_R(M, V)$ is a right Ω-module by $(\varphi(ag))(m) = g^{-1}\varphi(a(gm))$. It is easy to see that the standard isomorphism

$$\operatorname{Hom}_R(M \otimes_R V, W) \to \operatorname{Hom}_R(M, \operatorname{Hom}_R(V, W))$$

is an isomorphism of right Ω-modules for a left Ω-module M and G-modules V and W.

(3.14) Now consider the case $\Lambda = R$. Then the pairing $\phi : RG \otimes_R RG \to R$ given by $\phi(g \otimes g') = \delta_{gg',e}$ (Kronecker's delta) is non-degenerate, and induces an RG-bimodule isomorphism $\Omega = RG \to (RG)^* = \Omega^*$. As $\Omega = RG$ is a finite free R-module, we have that $\Omega = RG$ is symmetric relative to R.

Lemma 3.15. *If Λ is quasi-symmetric (resp. symmetric) and the action of G on Λ is trivial, then Ω is quasi-symmetric (resp. symmetric).*

Proof. Taking the completion, we may assume that R is complete. Then replacing R by a Noether normalization of $R/\operatorname{ann}_R \Lambda$, we may assume that R is a regular local ring, and Λ is a faithful R-module. As the action of G on Λ is trivial, $\Omega = \Lambda \otimes_R RG$ is quasi-symmetric (resp. symmetric), as can be seen easily. $\square$

(3.16) In particular, if Λ is commutative quasi-Gorenstein (resp. Gorenstein) and the action of G on Λ is trivial, then $\Omega = \Lambda G$ is quasi-symmetric (resp. symmetric).

(3.17) In general, ${}_\Omega\Omega \cong \Lambda \otimes_R RG$ as Ω-modules.

Lemma 3.18. *Let M and N be right Ω-modules, and let $\varphi : M \to N$ be a homomorphism of right Λ-modules. Then $\psi : M \otimes RG \to N \otimes RG$ given by $\psi(m \otimes g) = g(\varphi(g^{-1}m)) \otimes g$ is an Ω-homomorphism. In particular,*

1 *If φ is a Λ-isomorphism, then ψ is an Ω-isomorphism.*
2 *If φ is a split monomorphism in $\operatorname{mod} \Lambda$, then ψ is a split monomorphism in $\operatorname{mod} \Omega$.*

Proof. Straightforward. $\square$

Proposition 3.19. *Let G be a finite group acting on Λ. Set $\Omega := \Lambda * G$.*

1 *If the action of G on Λ is trivial and Λ is quasi-symmetric (resp. symmetric), then so is Ω.*
2 *If Λ is pseudo-Frobenius (resp. Frobenius), then so is Ω.*
3 *If Λ is pseudo-quasi-Frobenius (resp. quasi-Frobenius), then so is Ω.*

Proof. **1** is Lemma 3.15. To prove **2** and **3**, we may assume that $(R, \mathfrak{m})$ is complete regular local and Λ is a faithful module.

2.

$$(K_\Omega)_\Omega \cong \operatorname{Hom}_R(\Lambda \otimes_R RG, R) \cong \operatorname{Hom}_R(\Lambda, R) \otimes (RG)^* \cong K_\Lambda \otimes RG$$

as right Ω-modules. It is isomorphic to $\Lambda_\Omega \otimes RG \cong \Omega_\Omega$ by Lemma 3.18, **1**, since $K_\Lambda \cong \Lambda$ in $\operatorname{mod} \Lambda$. Hence Ω is pseudo-Frobenius. If, in addition, Λ is Cohen–Macaulay, then Ω is also Cohen–Macaulay, and hence Ω is Frobenius.

3 is proved similarly, using Lemma 3.18, **2**. $\square$

Note that the assertions for Frobenius and quasi-Frobenius properties also follow easily from Lemma 3.9 and [7, (3.2)].

4. Codimension-two argument

(4.1) This section is the second part of this paper. In this section, we show that the codimension-two argument using the existence of 2-canonical modules in [4] is still valid in non-commutative settings, as announced in (1.3).

(4.2) Let X be a locally Noetherian scheme, U its open subscheme, and Λ a coherent $\mathcal{O}_X$-algebra. Let $i : U \hookrightarrow X$ be the inclusion.

(4.3) Let $\mathcal{M} \in \operatorname{mod} \Lambda$. That is, $\mathcal{M}$ is a coherent right Λ-module. Then by restriction, $i^*\mathcal{M} \in \operatorname{mod} i^*\Lambda$.

(4.4) For a quasi-coherent $i^*\Lambda$-module $\mathcal{N}$, we have an action

$$i_*\mathcal{N} \otimes_{\mathcal{O}_X} \Lambda \xrightarrow{1\otimes u} i_*\mathcal{N} \otimes_{\mathcal{O}_X} i_*i^*\Lambda \to i_*(\mathcal{N} \otimes_{\mathcal{O}_U} i^*\Lambda) \xrightarrow{a} i_*\mathcal{N},$$

where u is the unit map for the adjoint pair (i^*, i_*). So we get a functor $i_* : \operatorname{Mod} i^*\Lambda \to \operatorname{Mod} \Lambda$, where $\operatorname{Mod} i^*\Lambda$ (resp. $\operatorname{Mod} \Lambda$) denote the category of quasi-coherent $i^*\Lambda$-modules (resp. Λ-modules).

Lemma 4.5. *Let the notation be as above. Assume that U is large in X (that is, $\operatorname{codim}_X(X \setminus U) \geq 2$). If $\mathcal{M} \in (S_2')^{\Lambda^{\mathrm{op}},X}$, then the canonical map $u : \mathcal{M} \to i_*i^*\mathcal{M}$ is an isomorphism.*

Proof. Follows immediately from [4, (7.31)]. □

Proposition 4.6. *Let the notation be as above, and let U be large in X. Assume that there is a 2-canonical right Λ-module. Then we have the following.*

1 *If $\mathcal{N} \in (S_2')^{i^*\Lambda^{\mathrm{op}},U}$, then $i_*\mathcal{N} \in (S_2')^{\Lambda^{\mathrm{op}},X}$.*
2 *$i^* : (S_2')^{\Lambda^{\mathrm{op}},X} \to (S_2')^{i^*\Lambda^{\mathrm{op}},U}$ and $i_* : (S_2')^{i^*\Lambda^{\mathrm{op}},U} \to (S_2')^{\Lambda^{\mathrm{op}},X}$ are quasi-inverse each other.*

Proof. The question is local, and we may assume that X is affine.

1. There is a coherent subsheaf $\mathcal{Q}$ of $i_*\mathcal{N}$ such that $i^*\mathcal{Q} = i^*i_*\mathcal{N} = \mathcal{N}$ by [2, Exercise II.5.15]. Let $\mathcal{V}$ be the Λ-submodule of $i_*\mathcal{N}$ generated by $\mathcal{Q}$. That is, the image of the composite

$$\mathcal{Q} \otimes_{\mathcal{O}_X} \Lambda \to i_*\mathcal{N} \otimes_{\mathcal{O}_X} \Lambda \to i_*\mathcal{N}.$$

Note that $\mathcal{V}$ is coherent, and $i^*\mathcal{Q} \subset i^*\mathcal{V} \subset i^*i_*\mathcal{N} = i^*\mathcal{Q} = \mathcal{N}$.

Let $\mathcal{C}$ be a 2-canonical right Λ-module. Let $?^\dagger := \underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(?, \mathcal{C})$, $\Gamma = \underline{\operatorname{End}}_\Lambda \mathcal{C}$, and $?^\ddagger := \underline{\operatorname{Hom}}_\Gamma(?, \mathcal{C})$. Let $\mathcal{M}$ be the double dual $\mathcal{V}^{\dagger\ddagger}$. Then $\mathcal{M} \in (S_2')^{\Lambda^{\mathrm{op}},X}$, and hence

$$\mathcal{M} \cong i_*i^*\mathcal{M} \cong i_*i^*(\mathcal{V}^{\dagger\ddagger}) \cong i_*(i^*\mathcal{V})^{\dagger\ddagger} \cong i_*(\mathcal{N}^{\dagger\ddagger}) \cong i_*\mathcal{N}.$$

So $i_*\mathcal{N} \cong \mathcal{M}$ lies in $(S_2')^{\Lambda^{\mathrm{op}},X}$.

2 follows from **1** and Lemma 4.5 immediately. □

References

[1] S. Goto and K. Nishida, *Towards a theory of Bass numbers with applications to Gorenstein algebras*, Colloq. Math., **91** (2002), 191–253.

[2] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, **52**, Springer Verlag 1977.

[3] R. Hartshorne, *Generalized divisors on Gorenstein schemes*, K-Theory, **8** (1994), 287–339.

[4] M. Hashimoto, *Equivariant class group. III. Almost principal fiber bundles*, `arXiv:1503.02133v1`

[5] M. Hashimoto, *Canonical and n-canonical modules on a Noetherian algebra*, `arXiv:1508.07552v1`

[6] M. Hashimoto, *Canonical and n-canonical modules of a Noetherian algebra*, to appear in Nagoya Math. J.

[7] G. Scheja and U. Storch, *Quasi-Frobenius-Algebren und lokal vollständige Durchschnitte*, Manuscripta Math., **19** (1976), 75–104. Addendum: **20** (1977), 99–100.

[8] A. Skowroński and K. Yamagata, *Frobenius Algebras. I., Basic Representation Theory*, European Mathematical Society 2011.

Mitsuyasu Hashimoto
Department of Mathematics
Faculty of Science
Okayama University
Okayama, 700-8530 Japan
e-mail address: mh@okayama-u.ac.jp

(*Received June 10, 2016*)
(*Accepted August 8, 2016*)