F-rationality of the ring of modular invariants

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Dedicated to Professor Kei-ichi Watanabe

Abstract

Using the description of the Frobenius limit of modules over the ring of invariants under an action of a finite group on a polynomial ring over a field of characteristic $p > 0$ developed by Symonds and the author, we give a characterization of the ring of invariants with a positive dual F-signature. Combining this result and Kemper's result on depths of the ring of invariants under an action of a permutation group, we give an example of an F-rational, but non-F-regular ring of invariants under the action of a finite group.

1. Introduction

Study of ring theoretic properties of invariant subrings under the action of algebraic groups is an important aspect of invariant theory. In this paper, we study the F-rationality of invariant subrings under the action of finite groups on polynomial rings. The F-rational property, as well as F-regular property, is an important ring theoretic property in characteristic p, see section 2.

Let k be an algebraically closed field of characteristic $p > 0$. Let $V = k^d$, and G a finite subgroup of $GL(V)$ without psuedo-reflections. Let $B =$

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$\operatorname{Sym} V$, the symmetric algebra of V, and $A = B^G$. If the order $|G|$ of G is not divisible by p, then A is a direct summand subring of B, and is strongly F-regular, see Lemma 2.3.

Let p divide $|G|$. Broer [Bro] proved that A is not a direct summand subring of B hence A is not weakly F-regular (as A is not a splinter).

In this paper, we study when A is F-rational. In [Gla], Glassbrenner showed that the invariant subring $k[x_1, \ldots, x_n]^{A_n}$, where A_n is the alternating group which acts on the polynomial ring by permutation of variables, is Gorenstein F-pure but not F-rational when $p = \operatorname{char}(k) \geq 3$ and $n \equiv 0, 1 \mod p$. Later, Singh [Sin] proved that the same is true if $n \geq p \geq 3$. In this paper, we give an example of A which is F-rational, F-pure, but not F-regular. Apart from rings of the form $(\operatorname{Sym} V)^G$, there has long been known an example of two-dimensional F-rational F-pure non-F-regular ring [Hun, (3.13)]. In [Hun, (3.13)], an example of two-dimensional F-rational non-F-pure ring is also shown.

Sannai [San] defined the dual F-signature $s(M)$ of a finite module M over an F-finite local ring R of characteristic p. He proved that R is F-rational if and only if R is Cohen–Macaulay and the dual F-signature $s(\omega_R)$ of the canonical module ω_R of R is positive. Utilizing the description of the Frobenius limit of modules over $\hat{A}$ (the completion of A) by Symonds and the author, we give a characterization of V such that $s(\omega_{\hat{A}}) > 0$, see Theorem 5.4. The characterization is purely representation theoretic in the sense that the characterization depends only on the structure of B as a G-module, rather than a G-algebra.

Using the characterization and Kemper's result on the depth of the ring of invariants under the action of certain groups of permutations [Kem, (3.3)], we give an example of F-rational A for $p \geq 5$. We also get an example of A such that the dual F-signature $s_{\omega_{\hat{A}}}$ of the canonical module of the completion $\hat{A}$ is positive, but A (or equivalently, $\hat{A}$) is not Cohen–Macaulay. See Theorem 6.12.

In section 3, we introduce the *asymptotic surjective number* $\operatorname{asn}_N(M)$ for two finitely generated modules M and N ($N \neq 0$) over a Noetherian ring R, see Lemma 3.6. In section 4, using the definition and some basic results developed in section 3, we prove the formula $s(M) = \operatorname{asn}_M(\mathrm{FL}([M]))$, where FL denotes the Frobenius limit defined in [HasS]. Thus $s(M)$ depends only on $\mathrm{FL}([M])$. Using this, we give a characterization of a module M to have positive $s(M)$ in terms of $\mathrm{FL}([M])$ (Corollary 4.5).

Using this result and the description of the Frobenius limits of certain modules over $\hat{A}$ proved in [HasS], we give a characterization of V such that $s(\omega_{\hat{A}}) > 0$ in section 5. In section 6, we give the examples.

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2. Preliminaries

(2.1) Let p be a prime number, and R be a commutative ring of characteristic p. We denote the Frobenius map $R \to R$ given by $x \mapsto x^p$ by F_R or F. For $r \in \mathbb{Z}$, let ${}^rR$ be a copy of R. We consider that F^e is a map from ${}^rR$ to ${}^{r+e}R$. The element $a \in R$ viewed as an element of ${}^rR$ is denoted by ${}^ra$. So we have $F^e({}^ra) = {}^{r+e}(a^{p^e}) = ({}^{r+e}a)^{p^e}$. As F^e is a ring homomorphism, ${}^{r+e}R$ is an ${}^rR$-algebra through F^e. We understand that R stands for ${}^0R$.

For an R-module M and $r \in \mathbb{Z}$, the R-module M viewed as an ${}^rR$-module is denoted by ${}^rM$. Its element $m \in M$ viewed as an element of ${}^rM$ is denoted by ${}^rm$. For $e \geq 0$, ${}^eM$ is also an R-module through the Frobenius map $F^e : R \to {}^eR$. That is, the action of R on ${}^eM$ is given by $a{}^em = {}^e(a^{p^e}m)$.

We say that R is F-finite if ${}^1R$ is a finite R-module through the Frobenius map $F : R \to {}^1R$. Note that an F-finite Noetherian ring of characteristic p is excellent [Kun].

(2.2) Let R be an F-finite Noetherian ring of characteristic p. We say that R is *F-pure* if $F : R \to {}^1R$ is a split monomorphism as an R-linear map.

For $c \in R$ and $e \geq 0$, let $cF^e : R \to {}^eR$ be the map $x \mapsto {}^e(cx^{p^e})$. We say that R is *strongly F-regular* if for every nonzerodivisor c of R, there exists some $e \geq 1$ such that $cF^e : R \to {}^eR$ is a split monomorphism as an R-linear map. We say that R is *F-rational* if R is Cohen–Macaulay, and for every nonzerodivisor c of R, there exists some $e \geq 1$ such that $cF^e : R \to {}^eR$ is injective and its cokernel $M = \operatorname{Coker}(cF^e)$ is a maximal Cohen–Macaulay module, that is, for any maximal ideal $\mathfrak{m}$ of R, $\operatorname{depth} M_{\mathfrak{m}} = \dim R_{\mathfrak{m}}$ (this definition is equivalent to the one using tight closures, see [Vel]). Note that regular rings are strongly F-regular, strongly F-regular rings are F-rational, and F-rational rings are Cohen–Macaulay and normal, see [Hun].

Lemma 2.3. *Let S be an F-finite Noetherian commutative ring of characteristic p. Let G be a finite group acting on S. Let $R = S^G$ be the ring of invariants. Then we have:*

(1) *R is F-finite, and S is finite over R.*

(2) *Assume that the order $|G|$ of G is not divisible by p, and S is strongly F-regular. Then R is strongly F-regular.*

Proof. **(1)** is [Has2, (9.6)].

(2). Let $\rho = |G|^{-1} \sum_{g \in G} g$ be the Reynolds operator. Then $\rho : S \to R$ is the left inverse of the inclusion $R \hookrightarrow S$, and is an R-linear map. It follows that R is a pure subring of S, and hence by [Has, (3.17)], R is strongly F-regular. □

(2.4) Let $\mathcal{C}$ be an additive category. Its Grothendieck group $[\mathcal{C}]$ is defined by

$$[\mathcal{C}] := (\bigoplus_{M \in \operatorname{Iso} \mathcal{C}} \mathbb{Z} \cdot M)/(M - M_1 - M_2 \mid M \cong M_1 \oplus M_2),$$

where $\operatorname{Iso} \mathcal{C}$ is the set of isomorphism classes of objects in $\mathcal{C}$. The class of M in the group $[\mathcal{C}]$ is denoted by $[M]$. Let Γ be an abelian group. A map $f : \mathcal{C} \to \Gamma$ is called *additive* if $f(M) = f(M_1) + f(M_2)$ for $M, M_1, M_2 \in \mathcal{C}$ such that $M \cong M_1 \oplus M_2$. If so, then there is a unique homomorphism of abelian groups $f_* : [\mathcal{C}] \to \Gamma$ such that $f_*[M] = f(M)$ for $M \in \mathcal{C}$. An additive functor $h : \mathcal{C} \to \mathcal{D}$ yields $h_* : [\mathcal{C}] \to [\mathcal{D}]$ such that $h_*[M] = [hM]$.

(2.5) Let $(R, \mathfrak{m})$ be a Henselian local ring. Let $\mathcal{C} := \operatorname{mod} R$. As in [HasS], we define

$$[\mathcal{C}] := (\bigoplus_{M \in \mathcal{C}} \mathbb{Z} \cdot M)/(M - M_1 - M_2 \mid M \cong M_1 \oplus M_2),$$

and $[\mathcal{C}]_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} [\mathcal{C}]$. In [HasS], $[\mathcal{C}]_{\mathbb{R}}$ is also written as $\Theta^\wedge(R)$ or $\Theta(R)$ (considering that R is trivially graded). In this paper, we write it as $\Theta(R)$. For $M \in \mathcal{C}$, we denote by $[M]$ the class of M in $\Theta(R)$. For an isomorphism class N of modules, $[N]$ is a well-defined element of $\Theta(R)$. Let $\operatorname{Ind}(R)$ denote the set of isomorphism classes of indecomposable modules in $\mathcal{C}$. The set $[\operatorname{Ind}(R)] := \{[M] \mid M \in \operatorname{Ind}(R)\}$ is an $\mathbb{R}$-basis of $\Theta(R) = [\mathcal{C}]_{\mathbb{R}}$. So $\alpha \in \Theta(R)$ can be written $\alpha = \sum_{M \in \operatorname{Ind}(R)} c_M[M]$ with $c_M \in \mathbb{R}$ uniquely. We say that $\alpha \geq 0$ if $c_M \geq 0$ for every $M \in \operatorname{Ind}(R)$. For $\alpha, \beta \in \Theta(R)$, we define $\alpha \geq \beta$ if $\alpha - \beta \geq 0$. This gives a partial ordering on $\Theta(R)$.

(2.6) In [HasS], a norm on $\Theta(R)$ was considered. For $\alpha = \sum_{M \in \operatorname{Ind}(R)} c_M[M] \in \Theta(R)$, we define $\|\alpha\| := \sum_M |c_M| \mu_R(M)$, where $\mu_R(M)$ is the number of generators $\ell_R(M/\mathfrak{m}M)$ (ℓ_R denotes the length) of M. It is easy to see that $\| \ \|$ is a norm, and hence $\Theta(R)$ is a metric space.

(2.7) Let $(R, \mathfrak{m}, k)$ be a F-finite Henselian local Noetherian commutative ring of characteristic p. Let $d := \dim R$, $\mathfrak{d} := \log_p[k : k^p]$, and $\delta := d + \mathfrak{d}$. For $\alpha = \sum_{M \in \operatorname{Ind} R} c_M[M] \in \Theta(R)$ ($c_M \in \mathbb{R}$) and $e \geq 0$, we define ${}^e\alpha = \sum_{M \in \operatorname{Ind} R} c_M[{}^eM]$, and call it the eth Frobenius direct image of α. We define

$$\operatorname{FL}(\alpha) := \lim_{e \to \infty} \frac{1}{p^{\delta e}} {}^e\alpha,$$

provided the limit exists. We call $\operatorname{FL}(\alpha)$ the Frobenius limit of α, see [HasS].

3. Asymptotic surjective number

(3.1) Let R be a Noetherian commutative ring. Let $\operatorname{mod} R$ denote the category of finite R-modules.

(3.2) For $M, N \in \operatorname{mod} R$, we set

$$\operatorname{surj}_N^R(M) = \operatorname{surj}_N(M) := \sup\{n \in \mathbb{Z}_{\geq 0} \mid \text{There is a surjective } R\text{-linear map } M \to N^{\oplus n}\},$$

and call $\operatorname{surj}_N(M)$ the surjective number of M with respect to N. If $N = 0$, this is understood to be ∞.

Lemma 3.3. *Let $M, M', N \in \operatorname{mod} R$. Then we have the following.*

1 *If R' is any Noetherian R-algebra, then*

$$\operatorname{surj}_N^R(M) \leq \operatorname{surj}_{R' \otimes_R N}^{R'}(R' \otimes_R M).$$

2 *If $(R, \mathfrak{m})$ is local and $N \neq 0$, then $\operatorname{surj}_N^R(M) \leq \mu_R(M)/\mu_R(N)$, where $\mu_R = \ell_R(R/\mathfrak{m} \otimes_R -)$ denotes the number of generators.*

3 *If $N \neq 0$, then $\operatorname{surj}_N(M) < \infty$, and is a non-negative integer.*

4 *If $N \neq 0$, then $\operatorname{surj}_N(M) + \operatorname{surj}_N(M') \leq \operatorname{surj}_N(M \oplus M')$.*

5 *If* $N \neq 0$ *and* $r \geq 0$, *then* $r \operatorname{surj}_N(M) \leq \operatorname{surj}_N(M^{\oplus r})$.

Proof. **1** If there is a surjective R-linear map $M \to N^{\oplus n}$, then there is a surjective R'-linear map $R' \otimes_R M \to (R' \otimes_R N)^{\oplus n}$, and hence $n \leq \operatorname{surj}^{R'}_{R' \otimes_R N}(R' \otimes_R M)$.

2 By **1**, $\operatorname{surj}^R_N(M) \leq \operatorname{surj}^{R/\mathfrak{m}}_{N/\mathfrak{m}N}(M/\mathfrak{m}M) \leq \mu_R(M)/\mu_R(N)$ by dimension counting.

3 Take $\mathfrak{m} \in \operatorname{supp}_R N$. Then

$$\operatorname{surj}^R_N(M) \leq \operatorname{surj}^{R_\mathfrak{m}}_{N_\mathfrak{m}}(M_\mathfrak{m}) \leq \mu_{R_\mathfrak{m}}(M_\mathfrak{m})/\mu_{R_\mathfrak{m}}(N_\mathfrak{m}) < \infty.$$

4 Let $n = \operatorname{surj}_N(M)$ and $n' = \operatorname{surj}_N(M')$. Then there are surjective R-linear maps $M \to N^{\oplus n}$ and $M' \to N^{\oplus n'}$. Summing them, we get a surjective map $M \oplus M' \to N^{\oplus (n+n')}$.

5 follows from **4**. □

(3.4) Let $N, M \in \operatorname{mod} R$. Assume that N is nonzero. We define

$$\operatorname{nsurj}_N(M; r) := \frac{1}{r} \operatorname{surj}_N(M^{\oplus r})$$

for $r \geq 1$.

Lemma 3.5. *Let* $r \geq 1$, *and* $M, M', N \in \operatorname{mod} R$. *Assume that* $N \neq 0$. *Then*

1 $\operatorname{nsurj}_N(M; 1) = \operatorname{surj}_N(M)$.

2 $\operatorname{nsurj}_N(M; kr) \geq \operatorname{nsurj}_N(M; r)$ *for* $k \geq 0$.

3 $\operatorname{nsurj}_N(M; r) \geq \operatorname{surj}_N(M) \geq 0$.

4 $\operatorname{nsurj}_N(M; r) + \operatorname{nsurj}_N(M'; r) \leq \operatorname{nsurj}_N(M \oplus M'; r)$.

5 *If* $R \to R'$ *is a homomorphism of Noetherian rings, then* $\operatorname{nsurj}_N(M; r) \leq \operatorname{nsurj}_{R' \otimes_R N}(R' \otimes_R M; r)$.

6 *If* $(R, \mathfrak{m})$ *is local,* $\operatorname{nsurj}_N(M; r) \leq \mu_R(M)/\mu_R(N)$. *In general,* $\operatorname{nsurj}_N(M; r)$ *is bounded.*

Proof. **1** is by definition.

2. $kr \operatorname{nsurj}_N(M; kr) = \operatorname{surj}_N(M^{\oplus kr}) \geq k \operatorname{surj}_N(M^{\oplus r})$ by Lemma 3.3, **5**. Dividing by kr, we get the desired inequality.

3. This is immediate by **1** and **2**.

4 follows from Lemma 3.3, **4**.

5 follows from Lemma 3.3, **1**.

6 The first assertion is by Lemma 3.3, **2**. The second assertion follows from the first assertion and **5** applied to $R \to R' = R_{\mathfrak{m}}$, where $\mathfrak{m}$ is any element of $\operatorname{supp}_R N$. $\square$

Lemma 3.6. *Let $M, N \in \operatorname{mod} R$. Assume that $N \neq 0$. Then the limit*

$$\lim_{r\to\infty} \operatorname{nsurj}_N(M; r) = \lim_{r\to\infty} \frac{1}{r} \operatorname{surj}_N(M^{\oplus r})$$

exists.

We call the limit the *asymptotic surjective number of M with respect to N*, and denote it by $\operatorname{asn}_N(M)$.

Proof. As $\operatorname{nsurj}_N(M; r)$ is bounded, $S = \limsup_{r\to\infty} \operatorname{nsurj}_N(M; r)$ and $I = \liminf_{r\to\infty} \operatorname{nsurj}_N(M; r)$ exist. Assume for contradiction that the limit does not exist. Then $S > I$. Set $\varepsilon = (S - I)/2 > 0$.

There exists some $r_0 \geq 1$ such that $\operatorname{nsurj}_N(M; r_0) > S - \varepsilon/2$. Take $n_0 \geq 1$ sufficiently large so that $\operatorname{nsurj}_N(M; r_0)/n_0 < \varepsilon/2$. Let $r \geq r_0 n_0$, and set $n := \lfloor r/r_0 \rfloor$. Note that $nr_0 \leq r < (n + 1)r_0$ and $n \geq n_0$.

Then

$$\begin{aligned}\operatorname{nsurj}_N(M; r) &\geq \frac{1}{(n+1)r_0} \operatorname{surj}_N(M^{\oplus nr_0}) \geq \frac{n}{(n+1)r_0} \operatorname{surj}_N(M^{\oplus r_0}) \\ &= (1 - \frac{1}{n+1}) \operatorname{nsurj}_N(M; r_0) \geq \operatorname{nsurj}_N(M; r_0) - \varepsilon/2 > S - \varepsilon.\end{aligned}$$

Hence

$$I \geq \inf_{r \geq r_0 n_0} \operatorname{nsurj}_N(M; r) \geq S - \varepsilon > S - 2\varepsilon = I,$$

and this is a contradiction. $\square$

Lemma 3.7. *Let $M, M', N \in \operatorname{mod} R$, and $N \neq 0$. Then*

1 $\operatorname{asn}_N(M^{\oplus r}) = r \operatorname{asn}_N(M)$.

2 $0 \leq \operatorname{surj}_N(M) \leq \operatorname{nsurj}_N(M; r) \leq \operatorname{asn}_N(M)$ *for any $r \geq 1$.*

3 $\operatorname{asn}_N(M) + \operatorname{asn}_N(M') \leq \operatorname{asn}_N(M \oplus M')$.

Proof. **1**.

$$r^{-1}\operatorname{asn}_N(M^{\oplus r}) = \lim_{r'\to\infty}\frac{1}{rr'}\operatorname{surj}_N(M^{\oplus rr'}) = \operatorname{asn}_N(M).$$

2. $0 \le \operatorname{surj}_N(M) \le \operatorname{nsurj}_N(M;r)$ is Lemma 3.5, **3**. So taking the limit, $\operatorname{surj}_N(M) \le \operatorname{asn}_N(M)$. So $\operatorname{surj}_N(M^{\oplus r}) \le \operatorname{asn}_N(M^{\oplus r}) = r\operatorname{asn}_N(M)$. Dividing by r, $\operatorname{nsurj}_N(M;r) \le \operatorname{asn}_N(M)$. ☐

Lemma 3.8. *Let k be a field, and V a k-vector space, and $n \ge 0$. Assume that $\dim_k V \le n$. Let Γ be a set of subspaces of V such that $\sum_{U\in\Gamma} U = V$. Then there exist some $U_1, \dots, U_{n'} \in \Gamma$ with $n' \le n$ such that $U_1 + \cdots + U_{n'} = V$.*

Proof. Trivial. ☐

Lemma 3.9. *Let k be a field, V a k-vector space, and Γ a set of subspaces of V. Let W and W' be subspaces of V such that $W + W' = V$. Assume that $W' \subset \sum_{U\in\Gamma} U$. If $\dim_k W' \le n$, then there exist some $U_1, \dots, U_{n'} \in \Gamma$ with $n' \le n$ such that $W + U_1 + \cdots + U_{n'} = V$.*

Proof. Apply Lemma 3.8 to the vector space V/W. ☐

Lemma 3.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $M, M', N \in \operatorname{mod} R$ with $N \ne 0$. Then*

$$\operatorname{surj}_N(M') \le \operatorname{surj}_N(M \oplus M') - \operatorname{surj}_N(M) \le \mu_R(M').$$

Proof. The first inequality is Lemma 3.3, **4**. We prove the second inequality. Let $m = \operatorname{surj}_N(M \oplus M')$ and $n = \mu_R(M')$. There is a surjective map $\varphi : M \oplus M' \to N^{\oplus m}$. Let $N_i = N$ be the ith summand of $N^{\oplus m}$. Let the bar $\bar{\ }$ denote the functor $R/\mathfrak{m} \otimes_R -$. Set $V = \bar{N}^{\oplus m}$, $W = \bar{\varphi}(\bar{M})$, and $W' = \bar{\varphi}(\bar{M}')$. Then by Lemma 3.9, there exists some index set $I \subset \{1, 2, \dots, m\}$ such that $\#I \le n$ and $W + \sum_{i\in I}\bar{N}_i = V$. By Nakayama's lemma, $\varphi(M) + \sum_{i\in I} N_i = N^{\oplus m}$. This shows that

$$M \hookrightarrow M \oplus M' \xrightarrow{\varphi} N^{\oplus m} \to N^{\oplus m}/\sum_{i\in I} N_i \cong N^{\oplus(m-\#I)}$$

is surjective. Hence $\operatorname{surj}_N(M) \ge m - \#I \ge m - n$, and the result follows. ☐

(3.11) Let $(R, \mathfrak{m})$ be a Henselian local ring. For $\alpha = \sum_{M \in \operatorname{Ind}(R)} c_M[M] \in \Theta(R)$, we define

$$\langle \alpha \rangle := \sum_{M \in \operatorname{Ind}(R)} \max(0, \lfloor c_M \rfloor)[M].$$

So there exists some $M_\alpha \in \mathcal{C}$, unique up to isomorphisms, such that $\langle \alpha \rangle = [M_\alpha]$. For $N \in \operatorname{mod} R$ with $N \neq 0$, we define $\operatorname{surj}_N \alpha$ to be $\operatorname{surj}_N M_\alpha$.

(3.12) For $\alpha = \sum_{M \in \operatorname{Ind}(R)} c_M M \in \Theta(R)$, we define $\operatorname{supp} \alpha = \{M \in \operatorname{Ind}(R) \mid c_M > 0\}$. We define $Y(\alpha) = \bigoplus_{W \in \operatorname{supp} \alpha} W$ and $\nu(\alpha) := \mu_R(Y(\alpha))$.

Lemma 3.13. *Let $N \in \operatorname{mod} R$, $N \neq 0$, and $\alpha, \beta \in \Theta(R)$.*

1 *If $\alpha, \beta \geq 0$, then $0 \leq \operatorname{surj}_N \alpha \leq \operatorname{surj}_N(\alpha + \beta) - \operatorname{surj}_N \beta$.*

2 $|\operatorname{surj}_N \alpha - \operatorname{surj}_N \beta| \leq \|\alpha - \beta\| + \nu(\inf\{\alpha, \beta\})$.

Proof. **1**. As $\alpha, \beta \geq 0$, we have that $\langle \alpha \rangle + \langle \beta \rangle \leq \langle \alpha + \beta \rangle$. So by Lemma 3.3, **4**, $\operatorname{surj}_N \alpha + \operatorname{surj}_N \beta \leq \operatorname{surj}_N(\alpha + \beta)$.

2. Replacing α by $\sup\{\alpha, 0\}$ and β by $\sup\{\beta, 0\}$, we may assume that $\alpha, \beta \geq 0$. Moreover, replacing α by $\sup\{\alpha, \beta\}$ and β by $\inf\{\alpha, \beta\}$, we may assume that $\alpha \geq \beta$. As we have $\langle \alpha \rangle - \langle \beta \rangle \leq \alpha - \beta + [Y(\beta)]$, by Lemma 3.10 we have that

$$\begin{aligned} \operatorname{surj}_N \alpha - \operatorname{surj}_N \beta \leq \|\langle \alpha \rangle - \langle \beta \rangle\| &\leq \|\alpha - \beta + [Y(\beta)]\| \\ &\leq \|\alpha - \beta\| + \|[Y(\beta)]\| = \|\alpha - \beta\| + \nu(\beta). \end{aligned}$$

This is what we wanted to prove. □

Lemma 3.14. *The limit*

$$\lim_{t \to \infty} \frac{1}{t} \operatorname{surj}_N(t\alpha)$$

exists for $N \in \operatorname{mod} R$, $N \neq 0$ and $\alpha \in \Theta(R)$.

We denote the limit by $\operatorname{asn}_N(\alpha)$.

Proof. Replacing α by $\sup\{0, \alpha\}$, we may assume that $\alpha \geq 0$. Let $\varepsilon > 0$. We can take $W \in \operatorname{mod} R$ and an integer $n > 0$ such that $\alpha - n^{-1}[W] \geq 0$ and $\|\alpha - n^{-1}[W]\| < \varepsilon/8$. As $\operatorname{asn}_N W$ exists, there exists some $r_0 \geq 1$ such that for any $r \geq r_0$, $|\operatorname{nsurj}_N(W; r) - \operatorname{asn}_N W| < n\varepsilon/8$. Set $R :=$

$\max\{r_0 n, 16\mu_R(W)/\varepsilon, 8n\|\alpha\|/\varepsilon\}$. Let $t > R$. Let $r := \lfloor t/n \rfloor$. Then $0 \leq t - rn < n$ and $r \geq r_0$. We have

$$\begin{aligned}
&|t^{-1}\operatorname{surj}_N(t\alpha) - n^{-1}\operatorname{asn}_N W| \leq t^{-1}|\operatorname{surj}_N(t\alpha) - \operatorname{surj}_N(W^{\oplus r})| \\
&\quad + ((rn)^{-1} - t^{-1})\operatorname{surj}_N(W^{\oplus r}) + |(rn)^{-1}\operatorname{surj}_N(W^{\oplus r}) - n^{-1}\operatorname{asn}_N W| \\
&\qquad < t^{-1}\|t\alpha - r[W]\| + t^{-1}\mu_R(W) + (rt)^{-1}\mu_R(W^{\oplus r}) + \varepsilon/8 \\
&\qquad \leq (n/t)\|\alpha\| + (nr/t)\|\alpha - n^{-1}[W]\| + \varepsilon/16 + \varepsilon/16 + \varepsilon/8 \\
&\qquad\qquad < \varepsilon/8 + \varepsilon/8 + \varepsilon/16 + \varepsilon/16 + \varepsilon/8 = \varepsilon/2.
\end{aligned}$$

So for $t_1, t_2 > R$,

$$|t_1^{-1}\operatorname{surj}_N(t_1\alpha) - t_2^{-1}\operatorname{surj}_N(t_2\alpha)| < \varepsilon,$$

and $\lim_{t\to\infty} t^{-1}\operatorname{surj}_N(t\alpha)$ exists, as desired. □

Lemma 3.15. *Let $\alpha, \beta \in \Theta(R)$ and $N \in \operatorname{mod} R$ with $N \neq 0$.*

1 *For $k \geq 0$, we have $\operatorname{asn}_N(k\alpha) = k\operatorname{asn}_N(\alpha)$.*

2 *For $k \geq 0$, $0 \leq \operatorname{surj}_N(k\alpha) \leq k\operatorname{asn}_N(\alpha) \leq k\|\alpha\|/\mu_R(N)$.*

3 *If $\alpha, \beta \geq 0$, then $\operatorname{asn}_N(\alpha+\beta) \geq \operatorname{asn}_N(\alpha) + \operatorname{asn}_N(\beta)$.*

4 $|\operatorname{asn}_N(\alpha) - \operatorname{asn}_N(\beta)| \leq \|\alpha - \beta\|$.

5 asn_N *is continuous.*

Proof. **1**. If $k = 0$, then both-hand sides are zero, and the assertion is clear. So we may assume that $k > 0$. Then

$$\operatorname{asn}_N(k\alpha) = \lim_{t\to\infty}\frac{1}{t}\operatorname{surj}(tk\alpha) = k\lim_{t\to\infty}\frac{1}{tk}\operatorname{surj}(tk\alpha) = k\operatorname{asn}_N(\alpha).$$

2. We may assume that $k > 0$. By **1**, replacing $k\alpha$ by α, we may assume that $k = 1$. Replacing α by $\sup\{0, \alpha\}$, we may assume that $\alpha \geq 0$. For $n \geq 0$, $n\langle\alpha\rangle \leq \langle n\alpha\rangle$. Hence, $n\operatorname{surj}_N(\alpha) \leq \operatorname{surj}_N(n\langle\alpha\rangle) \leq \operatorname{surj}_N(n\alpha)$. So $\operatorname{surj}_N(\alpha) \leq n^{-1}\operatorname{surj}_N(n\alpha)$. Passing to the limit, $\operatorname{surj}_N(\alpha) \leq \operatorname{asn}_N(\alpha)$. Similarly,

$$\frac{1}{n}\operatorname{surj}_N(n\alpha) \leq \frac{\|\langle n\alpha\rangle\|}{n\mu_R(N)} \leq \frac{\|n\alpha\|}{n\mu_R(N)} = \frac{\|\alpha\|}{\mu_R(N)}.$$

Passing to the limit, $\operatorname{asn}_N(\alpha) \leq \frac{\|\alpha\|}{\mu_R(N)}$, as desired.

3. By Lemma 3.13, **1**, for $t > 0$,

$$\frac{1}{t}\operatorname{surj}_N(t\alpha) + \frac{1}{t}\operatorname{surj}_N(t\beta) \leq \frac{1}{t}\operatorname{surj}_N(t(\alpha+\beta)).$$

Passing to the limit, $\operatorname{asn}_N(\alpha) + \operatorname{asn}_N(\beta) \leq \operatorname{asn}_N(\alpha+\beta)$.

4. By Lemma 3.13, **2**,

$$\begin{aligned}|\frac{1}{t}\operatorname{surj}_N(t\alpha) - \frac{1}{t}\operatorname{surj}_N(t\beta)| \leq \frac{1}{t}(\|t(\alpha-\beta)\| + \nu(\inf\{t\alpha, t\beta\})) \\ = \|\alpha-\beta\| + \nu(\inf\{\alpha,\beta\})/t.\end{aligned}$$

Passing to the limit, $|\operatorname{asn}_N(\alpha) - \operatorname{asn}_N(\beta)| \leq \|\alpha-\beta\|$, as desired.

5 is an immediate consequence of **4**. □

4. Sannai's dual F-signature

(4.1) In this section, let p be a prime number, and $(R, \mathfrak{m}, k)$ be an F-finite local ring of characteristic p of dimension d. Let $\mathfrak{d} = \log_p[k : k^p]$, and $\delta = d + \mathfrak{d}$.

(4.2) In [San], for $M \in \operatorname{mod} R$, Sannai defined the dual F-signature of M by

$$s_R(M) = s(M) := \limsup_{e\to\infty} \frac{\operatorname{surj}_M({}^eM)}{p^{\delta e}}.$$

The dual F-signature $s(R)$ of R is the (usual) F-signature [HunL], which is closely related to the strong F-regularity of R [AL]. While $s(\omega_R)$ measures the F-rationality of R, provided R is Cohen–Macaulay.

Theorem 4.3 ([San, (3.16)])**.** *R is F-rational if and only if R is Cohen–Macaulay and $s(\omega_R) > 0$.*

Now we connect the Frobenius limit defined in [HasS] with dual F-signature.

Theorem 4.4. *Let R be Henselian, and $M \in \operatorname{mod} R$. Assume that the Frobenius limit*

$$\operatorname{FL}([M]) = \lim_{e\to\infty} \frac{1}{p^{\delta e}}[{}^eM] \in \Theta(R)$$

exists. Then

$$s_R(M) = \lim_{e\to\infty} \frac{\operatorname{surj}_M({}^eM)}{p^{\delta e}} = \operatorname{asn}_M(\operatorname{FL}([M])).$$

Proof. By Lemma 3.13,

$$p^{-\delta e}|\operatorname{surj}_M(p^{\delta e}\operatorname{FL}([M])) - \operatorname{surj}_M([{}^eM])| \\ \leq \|\operatorname{FL}([M]) - p^{-\delta e}[{}^eM]\| + p^{-\delta e}\nu(\operatorname{supp}(\operatorname{FL}([M]))).$$

Taking the limit $e \to \infty$, we get the desired result. □

Corollary 4.5. *Let the assumption be as in the theorem. Then the following are equivalent.*

1 $s(M) > 0$.

2 *For any* $N \in \operatorname{mod} R$ *such that* $\operatorname{supp}([N]) = \operatorname{supp}(\operatorname{FL}(M))$*, there exists some* $r \geq 1$ *and a surjective* R*-linear map* $N^{\oplus r} \to M$.

3 *There exist some* $N \in \operatorname{mod} R$ *such that* $\operatorname{supp}([N]) \subset \operatorname{supp}(\operatorname{FL}(M))$ *and a surjective* R*-linear map* $N \to M$.

Proof. **1**⇒**2**. As $\operatorname{asn}_M(\operatorname{FL}(M)) > 0$, there exists some $t > 0$ such that $\operatorname{surj}_M(t\operatorname{FL}(M)) > 0$. By the choice of N, there exists some $r \geq 1$ such that $r[N] \geq t\operatorname{FL}(M)$ and so $\operatorname{surj}_M N^{\oplus r} \geq \operatorname{surj}_M(t\operatorname{FL}(M)) > 0$.

2⇒**3**. Let $N = W_1 \oplus \cdots \oplus W_r$, where $\{W_1, \ldots, W_r\} = \operatorname{supp}(\operatorname{FL}(M))$. Then there exists some $r \geq 1$ and a surjective R-linear map $N^{\oplus r} \to M$, and $\operatorname{supp}[N^{\oplus r}] \subset \operatorname{supp}(\operatorname{FL}(M))$.

3⇒**1**. By the choice of N, there exists some $k > 0$ such that $k\operatorname{FL}(M) \geq [N]$. Then $s(M) = \operatorname{asn}_M(\operatorname{FL}(M)) \geq k^{-1}\operatorname{asn}_M[N] \geq k^{-1}\operatorname{surj}_M[N] > 0$. □

5. The dual F-signature of the ring of invariants

Utilizing the result in [HasS] and the last section, we give a criterion for the condition $s(\omega_{\hat{A}}) > 0$ for the ring of invariants A, where $\hat{A}$ is the completion.

(5.1) Let k be an algebraically closed field, $V = k^d$, G a finite subgroup of $GL(V)$. In this section, we assume that G does not have a pseudo-reflection, where we say that $g \in GL(V)$ is a pseudo-reflection if $\operatorname{rank}(g - 1_V) = 1$. Let $v_1, \ldots, v_d$ be a fixed k-basis of V. Let $B := \operatorname{Sym} V = k[v_1, \ldots, v_d]$, and $A = B^G$. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the irrelevant ideals of A and B, respectively. Let $\hat{A}$ and $\hat{B}$ be the completion of A and B, respectively.

For a G-module W, we define $M_W := (B \otimes_k W)^G$. Let $k = V_0, V_1, \ldots, V_n$ be the irreducible representations of G. Let $P_i \to V_i$ be the projective cover.

Set $M_i := M_{P_i} = (B \otimes_k P_i)^G$. For a finite dimensional G-module W, $\det_W$ denote the determinant representation $\bigwedge^{\dim W} W$ of W. Let $V_\nu = \det_V$ be the determinant representation of V.

Lemma 5.2. *The canonical module ω_A of A is isomorphic to $M_{\det_V}$.*

Proof. See [Has2, (14.28)] and references therein. □

Lemma 5.3. *Let Λ be a selfinjective finite dimensional k-algebra, L a simple (left) Λ-module, and $h : P \to L$ its projective cover. Let M be a finitely generated indecomposable Λ-module. Then the following are equivalent.*

1 $\operatorname{Ext}^1_\Lambda(M, \operatorname{rad} P) = 0$.

2 $h_* : \operatorname{Hom}_\Lambda(M, P) \to \operatorname{Hom}_\Lambda(M, L)$ *is surjective.*

3 *M is either projective, or $M/\operatorname{rad} M$ does not contain L.*

Proof. **1⇔2**. This is because

$$\operatorname{Hom}_\Lambda(M, P) \xrightarrow{h_*} \operatorname{Hom}_\Lambda(M, L) \to \operatorname{Ext}^1_\Lambda(M, \operatorname{rad} P) \to \operatorname{Ext}^1_\Lambda(M, P)$$

is exact and $\operatorname{Ext}^1_\Lambda(M, P) = 0$ (since P is injective).

2⇒3. Assume the contrary. Then as $M/\operatorname{rad} M$ contains L, there is a surjective map $M \to L$. By assumption, this map lifts to $M \to P$, and this is surjective by Nakayama's lemma. As P is projective, this map splits. As M is indecomposable, $M \cong P$, and this is a contradiction.

3⇒2. If M is projective, then h_* is obviously surjective. If $M/\operatorname{rad} M$ does not contain L, then $\operatorname{Hom}_\Lambda(M, L) = 0$, and h_* is obviously surjective. □

Theorem 5.4. *Let p divide the order $|G|$ of G. Then the following are equivalent.*

1 $s(\omega_{\hat{A}}) > 0$.

2 *The canonical map $M_\nu \to M_{V_\nu} = \omega_A$ is surjective.*

3 $H^1(G, B \otimes_k \operatorname{rad} P_\nu) = 0$.

4 *For any non-projective finitely generated indecomposable G-summand M of B, M does not contain $\det_V^{-1}$, the k-dual of $\det_V$.*

If these conditions hold, then $s(\omega_{\hat{A}}) \geq 1/|G|$.

Proof. We prove the equivalence of **2** and **3** first. Let $B = \bigoplus_j N_j$ be a decomposition into finitely generated indecomposable G-modules. Such a decomposition exists, since B is a direct sum of finitely generated G-modules. The map $M_\nu \to M_{V_\nu}$ in **2** is the map

$$(B \otimes P_\nu)^G \to (B \otimes \det_V)^G$$

induced by the projective cover $P_\nu \to \det_V$. By the isomorphism $\operatorname{Ext}^i_G(N_j^*, -) \cong H^i(G, N_j \otimes -)$, this map can be identified with the sum of

$$\operatorname{Hom}_G(N_j^*, P_\nu) \to \operatorname{Hom}_G(N_j^*, \det_V).$$

On the other hand, **3** is equivalent to say that $\operatorname{Ext}^1_G(N_j^*, \operatorname{rad} P_\nu) = 0$ for any j. So the equivalence **2**⇔**3** follows from Lemma 5.3.

Similarly, **4** is equivalent to say that each N_j^* is injective (or equivalently, projective, as kG is selfinjective) or $N_j^* / \operatorname{rad} N_j^* \cong (\operatorname{soc} N_j)^*$ does not contain $\det_V$. This is equivalent to say that N_j is either projective, or N_j (or equivalently, $\operatorname{soc} N_j$) does not contain $\det^{-1}$. So **4**⇔**2** follows from Lemma 5.3.

We prove **2**⇒**1**. As there is a surjective map $M_\nu \to \omega_A$ and

$$\operatorname{FL}([\omega_{\hat{A}}]) = \frac{1}{|G|} \sum_{i=0}^{n} (\dim V_i)[\hat{M}_i]$$

by [HasS, (5.1)], $s(\omega_{\hat{A}}) > 0$ by Corollary 4.5. Moreover,

$$s(\omega_{\hat{A}}) = \operatorname{asn}_{\omega_{\hat{A}}}(\operatorname{FL}([\omega_{\hat{A}}])) \geq \frac{\dim V_\nu}{|G|} \operatorname{asn}_{\omega_{\hat{A}}}(\hat{M}_\nu) \geq \frac{1}{|G|} \operatorname{surj}_{\omega_A}(M_\nu) \geq \frac{1}{|G|},$$

and the last assertion has been proved.

We prove **1**⇒**2**. By [HasS, (4.16)],

$$\operatorname{FL}([\omega_{\hat{A}}]) = \frac{1}{|G|}[\hat{B}].$$

So by Corollary 4.5, there is some $r > 0$ and a surjective map $h : \hat{B}^r \to \omega_{\hat{A}}$. By the equivalence $\gamma = (\hat{B} \otimes_{\hat{A}} -)^{**} : \operatorname{Ref}(\hat{A}) \to \operatorname{Ref}(G, \hat{B})$ (see [HasN, (2.4)] and [HasS, (5.4)]), there corresponds

$$\tilde{h} = \gamma(h) : (\hat{B} \otimes_k kG)^r \to \hat{B} \otimes_k \det.$$

As $\hat{B} \otimes_k kG$ is a projective object in the category of $(G, \hat{B})$-modules, $\tilde{h}$ factors through the surjection

$$\hat{B} \otimes_k P_\nu \to \hat{B} \otimes_k \det .$$

Returning to the category Ref $\hat{A}$, h factors through $\hat{M}_\nu = (\hat{B} \otimes_{\hat{A}} P_\nu)^G \to \omega_{\hat{A}}$. So this map must be surjective, and **2** follows. □

Corollary 5.5. *Assume that p divides $|G|$. If $s(\omega_{\hat{A}}) > 0$, then $\det_V^{-1}$ is not a direct summand of B.*

Proof. Being a one-dimensional representation, $\det_V^{-1}$ is not projective by assumption. Thus the result follows from **1⇒4** of the theorem. □

Lemma 5.6. *Let M and N be in* $\mathrm{Ref}(G, B)$. *There is a natural isomorphism*

$$\gamma : \mathrm{Hom}_A(M^G, N^G) \to \mathrm{Hom}_B(M, N)^G.$$

Proof. This is simply because $\gamma = (B \otimes_A -)^{**} : \mathrm{Ref}(A) \to \mathrm{Ref}(G, B)$ is an equivalence, and $\mathrm{Hom}_B(M, N)^G = \mathrm{Hom}_{G,B}(M, N)$. □

Theorem 5.7. *A is F-rational if and only if the following three conditions hold.*

1 *A is Cohen–Macaulay.*

2 $H^1(G, B) = 0$.

3 *$(B \otimes_k (I/k))^G$ is a maximal Cohen–Macaulay A-module, where I is the injective hull of k.*

Proof. If the order $|G|$ of G is not divisible by p, then A is F-rational, and the three conditions hold. So we may assume that $|G|$ is divisible by p.

Assume that A is F-rational. Then A is Cohen–Macaulay. As $s(\omega_{\hat{A}}) > 0$, we have that $H^1(G, B \otimes_k \mathrm{rad}\, P_\nu) = 0$, and

$$0 \to (B \otimes \mathrm{rad}\, P_\nu)^G \to (B \otimes P_\nu)^G \to (B \otimes \det_V)^G \to 0 \tag{1}$$

is exact. As $M_\nu = (B \otimes P_\nu)^G$ is a direct summand of $B = M_{kG} = (B \otimes kG)^G$, it is a maximal Cohen–Macaulay module. As $(B \otimes \det)^G = \omega_A$, it is also a maximal Cohen–Macaulay module. So the canonical dual of the exact sequence (1) is still exact. As there is an identification

$$\mathrm{Hom}_A((B \otimes_k -)^G, \omega_A) = \mathrm{Hom}_B(B \otimes_k -, B \otimes_k \det_V)^G = (B \otimes_k -^* \otimes_k \det_V)^G,$$

we get the exact sequence of maximal Cohen–Macaulay A-modules

(2) $\quad 0 \to A \to (B \otimes_k P_\nu^* \otimes_k \det_V)^G \to (B \otimes_k (\operatorname{rad} P_\nu)^* \otimes_k \det_V)^G \to 0.$

As $(\operatorname{rad} P_\nu)^* \otimes \det_V \cong I/k$, $(B \otimes (I/k))^G$ is maximal Cohen–Macaulay. As I is an injective G-module, $B \otimes_k I$ is so as a G-module, and hence $H^1(G, B \otimes_k I) = 0$. By the long exact sequence of the G-cohomology, we get $H^1(G, B) = 0$.

The converse is similar. Dualizing (2), we have that (1) is exact. $\square$

Corollary 5.8. *If A is F-rational, then $H^1(G, k) = 0$.*

Proof. k is a direct summand of B, and $H^1(G, B) = 0$. $\square$

Example 5.9. If $p = 2$ and $G = S_2$ or S_3, the symmetric groups, then $H^1(G, k) \neq 0$. So A is not F-rational, provided G does not have a pseudo-reflection.

6. An example of F-rational ring of invariants which are not F-regular

(6.1) Let p be an odd prime number, and k an algebraically closed field of characteristic p.

(6.2) Let us identify $\operatorname{Map}(\mathbb{F}_p, \mathbb{F}_p)^\times$ with the symmetric group S_p. We write $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$. Define

$$\begin{aligned} G &:= \{\phi \in S_p \mid \exists a \in \mathbb{F}_p^\times\ \exists b \in \mathbb{F}_p\ \forall x \in \mathbb{F}_p\ \phi(x) = ax + b\} \subset S_p; \\ Q &:= \{\phi \in S_p \mid \exists b \in \mathbb{F}_p\ \forall x \in \mathbb{F}_p\ \phi(x) = x + b\} \subset G; \\ \Gamma &:= \{\phi \in S_p \mid \exists a \in \mathbb{F}_p^\times\ \forall x \in \mathbb{F}_p\ \phi(x) = ax\} \subset G. \end{aligned}$$

G is a subgroup of S_p, Q is a normal subgroup of G, and Γ is a subgroup of G such that $G = Q \rtimes \Gamma$. Note that Q is cyclic of order p. Γ is cyclic of order $p-1$. So G is of order $p(p-1)$.

(6.3) Let α be a primitive element of $\mathbb{F}_p$ (that is, a generator of the cyclic group $\mathbb{F}_p^\times$), and let $\tau \in \Gamma$ be the element given by $\tau(x) = \alpha x$. The only involution of Γ is $\tau^{(p-1)/2}$, the multiplication by -1. As a permutation, it is

$$(1\ (p-1))(2\ (p-2)) \cdots ((p-1)/2\ (p+1)/2),$$

which is a transposition if and only if $p = 3$. As Γ contains a Sylow 2-subgroup, a transposition of G, if any, is conjugate to an element of Γ, and it must be a transposition again. It follows that G has a transposition if and only if $p = 3$.

(6.4) Now let $G \subset S_p$ act on $P = k^p = \langle w_0, w_1, \dots, w_{p-1} \rangle$ by the permutation action, that is, $\phi w_i = w_{\phi(i)}$ for $\phi \in G$ and $i \in \mathbb{F}_p$. $g \in G \subset GL(P)$ is a pseudo-reflection if and only if it is a transposition. So G has a pseudo-reflection if and only if $p = 3$.

Let $r \geq 1$, and set $V = P^{\oplus r}$. $G \subset GL(V)$ has a pseudo-reflection if and only if $p = 3$ and $r = 1$.

(6.5) Let $S = \operatorname{Sym} P$.

Lemma 6.6. *Let M be any finitely generated non-projective indecomposable G-summand of S. Then $M \cong k$.*

Proof. Let $\Omega = \{w^\lambda = w_0^{\lambda_0} \cdots w_{p-1}^{\lambda_{p-1}} \mid \lambda = (\lambda_0, \dots, \lambda_{p-1}) \in \mathbb{Z}^p_{\geq 0}\}$ be the set of monomials of S. G acts on the set Ω. Let Θ be the set of orbits of this action of G on Ω. Let $Gw^\lambda \in \Theta$.

If $\lambda = (r, r, \dots, r)$ for some $r \geq 0$, then $Gw^\lambda = \{w^\lambda\}$, and hence $(kG)w^\lambda \cong k$.

Otherwise, Q does not have a fixed point on the action on Gw^λ. As the order of Q is p, Q acts freely on Gw^λ. Hence $(kG)w^\lambda$ is kQ-free.

Since the order of $G/Q \cong \Gamma$ is $p-1$, the Lyndon–Hochschild–Serre spectral sequence collapses, and we have $H^i(G, M) \cong H^i(Q, M)^\Gamma$ for any G-module M. So a Q-injective (or equivalently, Q-projective) G-module is G-injective (or equivalently, G-projective).

As we have $S = \bigoplus_{\theta \in \Theta} k\theta$ as a G-module, S is a direct sum of G-projective modules and copies of k. Using Krull-Schmidt theorem, it is easy to see that $M \cong k$. □

Lemma 6.7. *Let U and W be G-modules.*

1 *$kG \otimes_k W \cong kG \otimes_k W'$, where W' is the k-vector space W with the trivial G-action.*

2 *If U is G-projective, then $U \otimes_k W$ is G-projective.*

Proof. **1**. $g \otimes w \mapsto g \otimes g^{-1} w$ gives such an isomorphism.

2 follows from **1**. □

(6.8) Let $B := \operatorname{Sym} V = \operatorname{Sym} P^{\oplus r} \cong S^{\otimes r}$.

Lemma 6.9. *Let M be any finitely generated non-projective indecomposable G-summand of B. Then $M \cong k$.*

Proof. Follows immediately from Lemma 6.6 and Lemma 6.7. □

Lemma 6.10. *Let k_- denote the sign representation. Then $\det_V \cong k_-$ if r is odd, and $\det_V \cong k$ if r is even. k_- is not isomorphic to k.*

Proof. As the determinant of a sign matrix is the signature of the permutation, $\det_P \cong k_-$. Hence $\det_V \cong (\det_P)^{\otimes r} \cong (k_-)^{\otimes r}$, and we get the desired result. The last assertion is clear, since $\tau = (x \mapsto \alpha x) \in \Gamma$ is a cyclic permutation of order $p-1$, and is an odd permutation. □

Theorem 6.11. *We have*

$$\operatorname{depth} A = \min\{rp, 2(p-1)+r\}.$$

Hence A is Cohen–Macaulay if and only if $r \leq 2$.

Proof. This is an immediate consequence of [Kem, (3.3)]. □

Theorem 6.12. *Let p, r, G, P, $V = P^{\oplus r}$, $B = \operatorname{Sym} V$ be as above, and $A := B^G$. Then*

1 *G is a finite subgroup of $GL(V)$ of order $p(p-1)$.*

2 *$G \subset GL(V)$ has a pseudo-reflection if and only if $p = 3$ and $r = 1$. If so, $G = S_3$ is the symmetric group acting regularly on $B = k[w_0, w_1, w_2]$ by permutations on w_0, w_1, w_2. The ring of invariants A is the polynomial ring. Otherwise, A is not weakly F-regular.*

3 *If $p \geq 5$ and $r = 1$, then A is F-rational, but not weakly F-regular.*

4 *If $r = 2$, then A is Gorenstein, but not F-rational.*

5 *If $r \geq 3$ and r is odd, then $s(\omega_{\hat{A}}) > 0$ but A is not Cohen–Macaulay.*

6 *If $r \geq 4$ and even, then A is quasi-Gorenstein, but not Cohen–Macaulay.*

7 *In any case, A is F-pure.*

Proof. We have already seen **1** and the first statement of **2**. If $p = 3$ and $r = 1$, then $G \subset S_3$ has order 6, and $G = S_3$. So A is the polynomial ring generated by the symmetric polynomials. Otherwise, as G does not have a pseudo-reflection and the order $|G|$ of G is divisible by p, A is not weakly F-regular, see [Bro], [Yas], and [HasS, (5.8)].

The only non-projective finitely generated indecomposable G-summand of B is k by Lemma 6.9, and $\det_V^{-1} \subset k$ if and only if r is even by Lemma 6.10. Hence we have that $s(\omega_{\hat{A}}) > 0$ if and only if r is odd by Theorem 5.4.

3. A is not weakly F-regular by **2**. As $r = 1$ is odd, $s(\omega_{\hat{A}}) > 0$. On the other hand, A is Cohen–Macaulay by Theorem 6.11. Hence A is F-rational by Theorem 4.3.

4. By Theorem 6.11, A is Cohen–Macaulay. On the other hand, by Lemma 6.10, $\det_V \cong k$, and hence $\omega_A \cong (B \otimes_k \det_V)^G \cong B^G \cong A$ by Lemma 5.2. So A is Gorenstein. As A is Gorenstein but not weakly F-regular, it is not F-rational by [HH, (4.7)].

5 and **6** are easy.

7 is an immediate consequence of [Gla, (2.4)]. □

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